Unit 6

Solution of Partial Differential Equation

A partial differential equation is a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variable), and a partial derivatives of the unknown function with respect to the independent variables.

If we represent the dependent variable as f and the two independent variables as x & y then second-order equation is given as

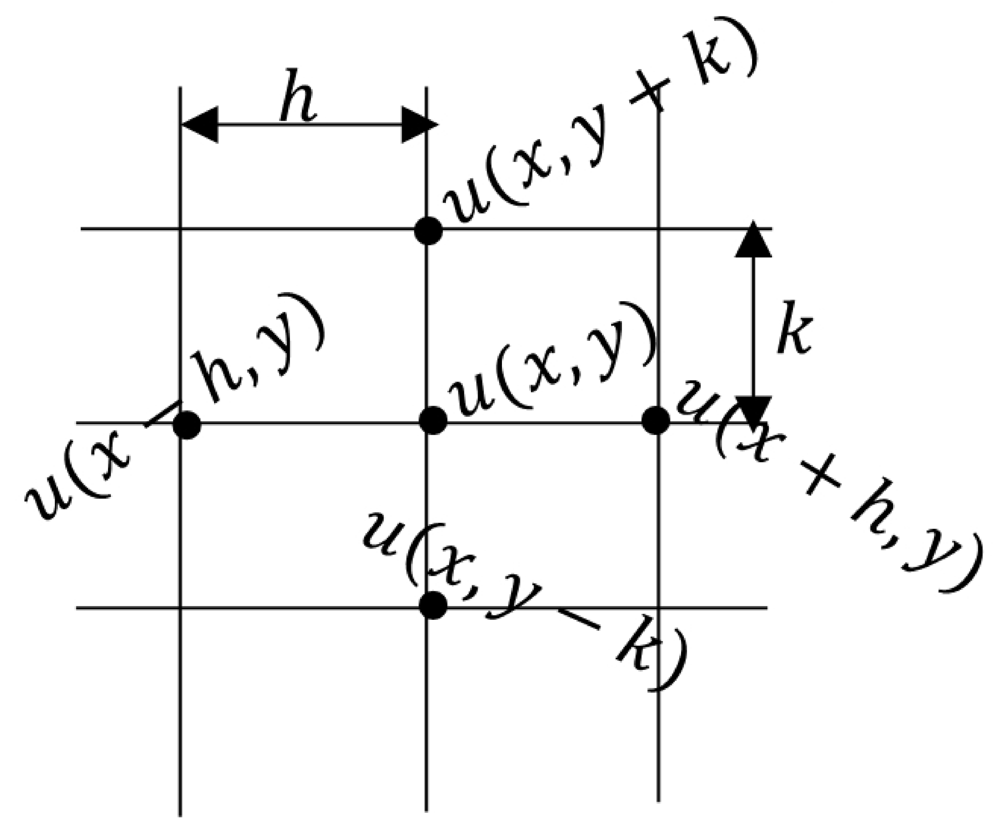
$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y^2} = F(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$$

Where the coefficients a, b & c may be constants or functions of x & y. Depending upon the value of $b^2 - 4ac$, a 2nd order linear partial differential equation can be classified into three categories:

- Elliptical at point (x, y) if $b^2 - 4ac < 0$
- Parabolic if $b^2 - 4ac = 0$
- Hyperbolic if $b^2 - 4ac > 0$

Deriving Difference Equations

Consider a two dimensional solution domain as shown in figure below;



The domain is split into regular rectangle grids of width h and height k.

Let $u(x, y)$ be the function of two independent variables x & y. Then by Taylor's formula

$$u(x+h, y) = u(x, y) + hu_x(x, y) + \frac{h^2}{2!}u_{xx}(x, y) + \frac{h^3}{3!}u_{xxx}(x, y) + \cdots \quad \text{(i)}$$

$$u(x-h, y) = u(x, y) - hu_x(x, y) + \frac{h^2}{2!}u_{xx}(x, y) - \frac{h^3}{3!}u_{xxx}(x, y) + \cdots \quad \text{(ii)}$$

$$u(x, y+k) = u(x, y) + ku_y(x, y) + \frac{k^2}{2!}u_{yy}(x, y) + \frac{k^3}{3!}u_{yyy}(x, y) + \cdots \quad \text{(iii)}$$

$$u(x, y-k) = u(x, y) - ku_y(x, y) + \frac{k^2}{2!}u_{yy}(x, y) - \frac{k^3}{3!}u_{yyy}(x, y) + \cdots \quad \text{(iv)}$$

Subtracting eq.(ii) by (i) & neglecting higher power of h we get;

$$u(x+h,y) - u(x-h,y) = 2hu_x(x,y)$$

$$\therefore u_x(x,y) = \frac{u(x+h,y) - u(x-h,y)}{2h}$$

Subtracting eq.(iv) by (iii) & neglecting higher power of k we get;

$$u(x,y+k) - u(x,y-k) = 2ku_y(x,y)$$

$$\therefore u_y(x,y) = \frac{u(x,y+k) - u(x,y-k)}{2k}$$

Adding eq.(i) and (ii) and neglecting higher of power of h we get;

$$u(x+h,y) + u(x-h,y) = 2u(x,y) + h^2u_{xx}(x,y)$$

$$\therefore u_{xx}(x,y) = \frac{1}{h^2}[u(x+h,y) - 2u(x,y) + u(x-h,y)] \quad \text{(a)}$$

Adding eq.(iii) and (iv) and neglecting higher of power of k we get;

$$u(x,y+k) + u(x,y-k) = 2u(x,y) + k^2u_{yy}(x,y)$$

$$\therefore u_{yy}(x,y) = \frac{1}{k^2}[u(x,y+k) - 2u(x,y) + u(x,y-k)] \quad \text{(b)}$$

Also,

$$u_{xy}(x,y) = \frac{u(x+h,y+k) - u(x+h,y-k) - u(x-h,y+k) + u(x-h,y-k)}{4hk}$$

Laplace's Equation

The equation $\boldsymbol{u_{xx} + u_{yy} = 0}$ is the Laplace equation, then from above eq.(a) & (b) we have,

$$\frac{1}{h^2}[u(x+h,y) - 2u(x,y) + u(x-h,y)] + \frac{1}{k^2}[u(x,y+k) - 2u(x,y) + u(x,y-k)] = 0$$

If $h = k$ we get,

$$u(x+h,y) + u(x,y+k) + u(x-h,y) + u(x,y-k) - 4u(x,y) = 0$$

$$\therefore\ u(x,y) = \frac{1}{4}[u(x+h,y) + u(x,y+k) + u(x-h,y) + u(x,y-k)]$$

This is the difference equation for Laplace's equation.

Poisson's Equation

The equation $\boldsymbol{u_{xx} + u_{yy} = g(x,y)}$ is the given Poisson's equation, then from above eq.(a) & (b) we have,

$$\frac{1}{h^2}[u(x+h,y) - 2u(x,y) + u(x-h,y)] + \frac{1}{k^2}[u(x,y+k) - 2u(x,y) + u(x,y-k)] = g(x,y)$$

If $h = k$ we get,

$$u(x+h,y) + u(x,y+k) + u(x-h,y) + u(x,y-k) - 4u(x,y) = h^2 g(x,y)$$

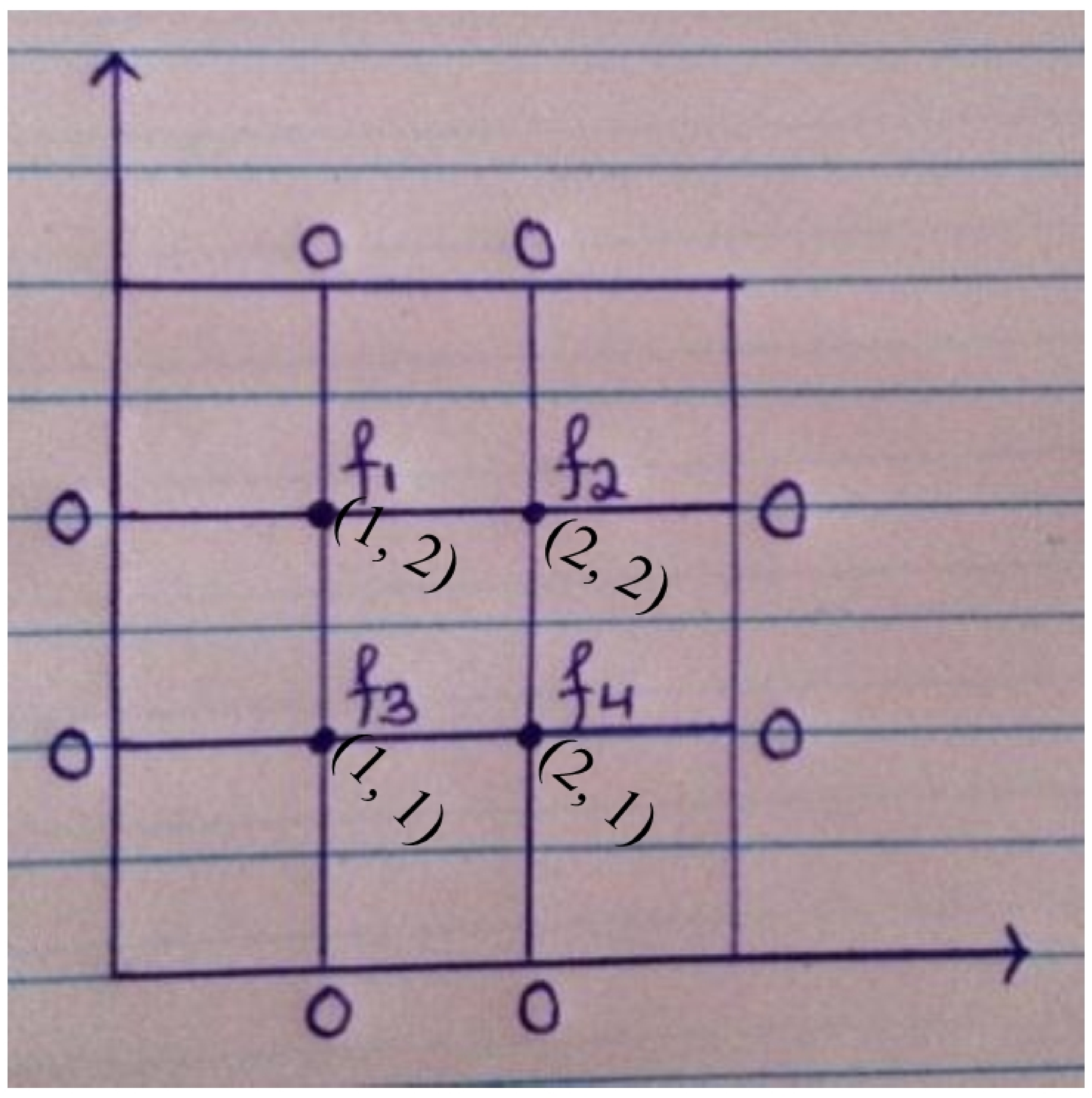
This is the difference equation for Poisson's equation.

Examples

1. Solve the Poisson's equation $\nabla^2 f = 2x^2y^2$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with $f = 0$ on the boundary and h=1.

Solution:

Given Poisson's eqn. is $\nabla^2 f = 2x^2y^2$; $0 \le x \le 3$, $0 \le y \le 3$
$h = 1$
Let's divide the domain into grids of 3×3 with $f = 0$ at the boundary as below



Now, from the difference equation for the Poisson's equation,

At f_1

$0 + 0 + f_2 + f_3 - 4f_1 = 1^2 \times 2 \times 1^2 \times 2^2$

Or, $f_2 + f_3 - 4f_1 = 8$ …….. (i)

At f_2

$0 + 0 + f_1 + f_4 - 4f_2 = 1^2 \times 2 \times 2^2 \times 2^2$

Or, $f_1 + f_4 - 4f_2 = 32$ …….. (ii)

At f_3

$0 + 0 + f_1 + f_4 - 4f_3 = 1^2 \times 2 \times 2^2 \times 1^2$

Or, $f_1 + f_4 - 4f_3 = 2$ …….. (iii)

At f_4

$0 + 0 + f_2 + f_3 - 4f_4 = 1^2 \times 2 \times 1^2 \times 1^2$

Or, $f_2 + f_3 - 4f_4 = 8$ …….. (iv)

Solving these equations,

Using eq.(ii) in (iii)

$32 - f_4 + 4f_2 + f_4 - 4f_3 = 2$

$4f_2 - 4f_3 = -30$ …………. (a)

Using eq.(ii) in (iv)

$f_2 + f_3 - 4(32 - f_1 + 4f_2) = 8$

$f_2 + f_3 - 128 + 4f_1 - 16f_2 = 8$

$-15f_2 + f_3 + 4f_1 = 136$ ………. (b)

& we have eq. (i)

$f_2 + f_3 - 4f_1 = 8$ …….. (i)

Solving equation (a), (b) & (i) we get,

$f_2 = -\frac{43}{4}$

$f_3 = -\frac{13}{4}$

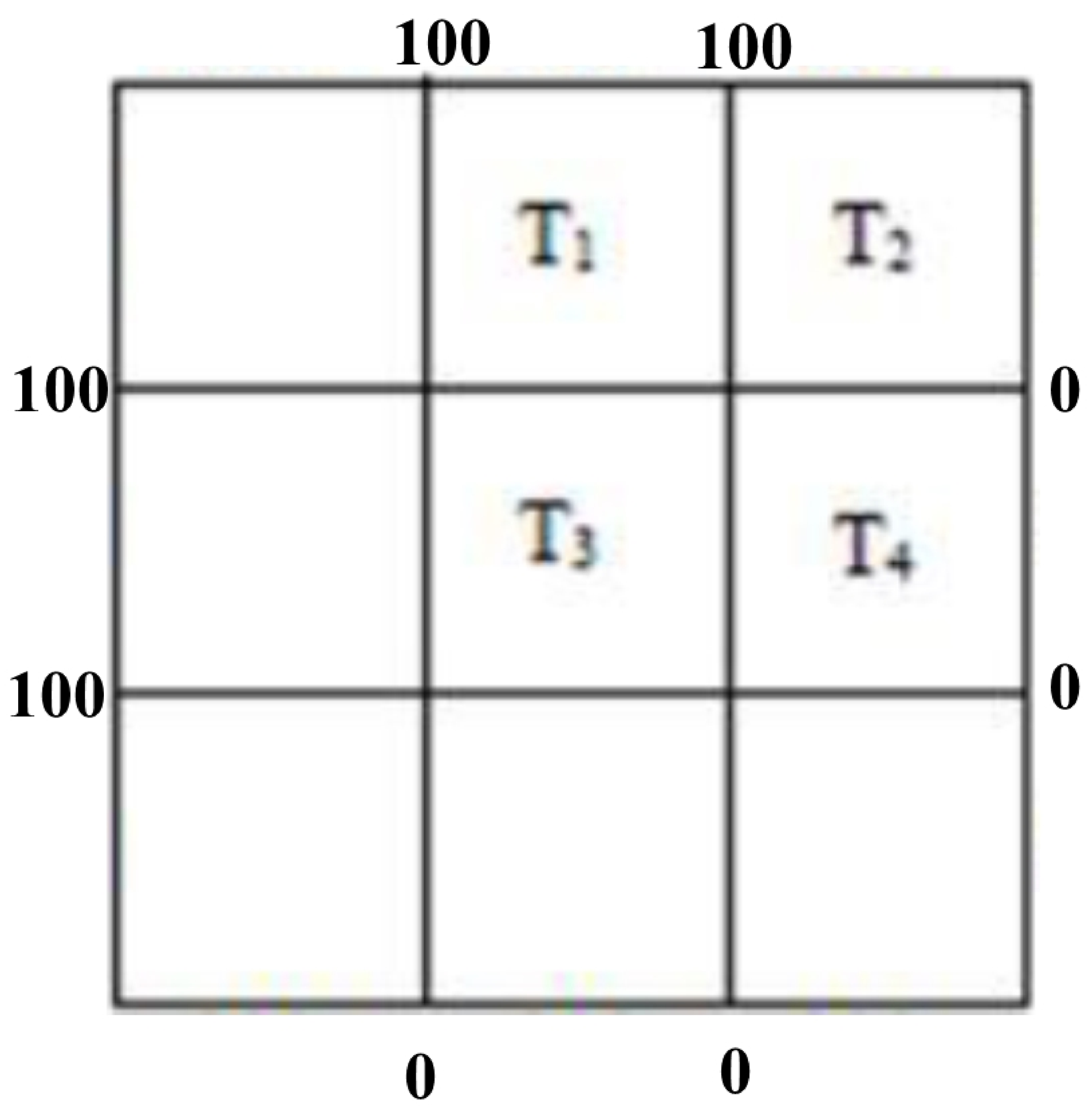
$f_1 = -\frac{11}{2}$

Using these values in eq. (ii)

$f_4 = -\frac{11}{2}$

$\therefore\ f_1 = -\frac{11}{2}, f_2 = -\frac{43}{4}, f_3 = -\frac{13}{4}\ \&\ f_4 = -\frac{11}{2}$

2. The steady state two dimensional heat-flow in a metal plate is defined by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$. Given the boundary conditions as sown in figure below, find the temperatures at interior points T_1, T_2, T_3 and T_4.



Solution:

Using the difference equation for the Laplace equation,

At point T_1

$$T_1 = \frac{1}{4}(100 + 100 + T_2 + T_3)$$

Or, $4T_1 - T_2 - T_3 = 200$ ……………. (i)

At point T_2

$$T_2 = \frac{1}{4}(100 + 0 + T_1 + T_4)$$

Or, $4T_2 - T_1 - T_4 = 100$ ……………. (ii)

At point T_3

$$T_3 = \frac{1}{4}(100 + 0 + T_1 + T_4)$$

Or, $4T_3 - T_1 - T_4 = 100$ .... (iii)

<u>At point T_4</u>

$$T_4 = \frac{1}{4}(0 + 0 + T_2 + T_3)$$

Or, $4T_4 - T_2 - T_3 = 0$ .... (iv)

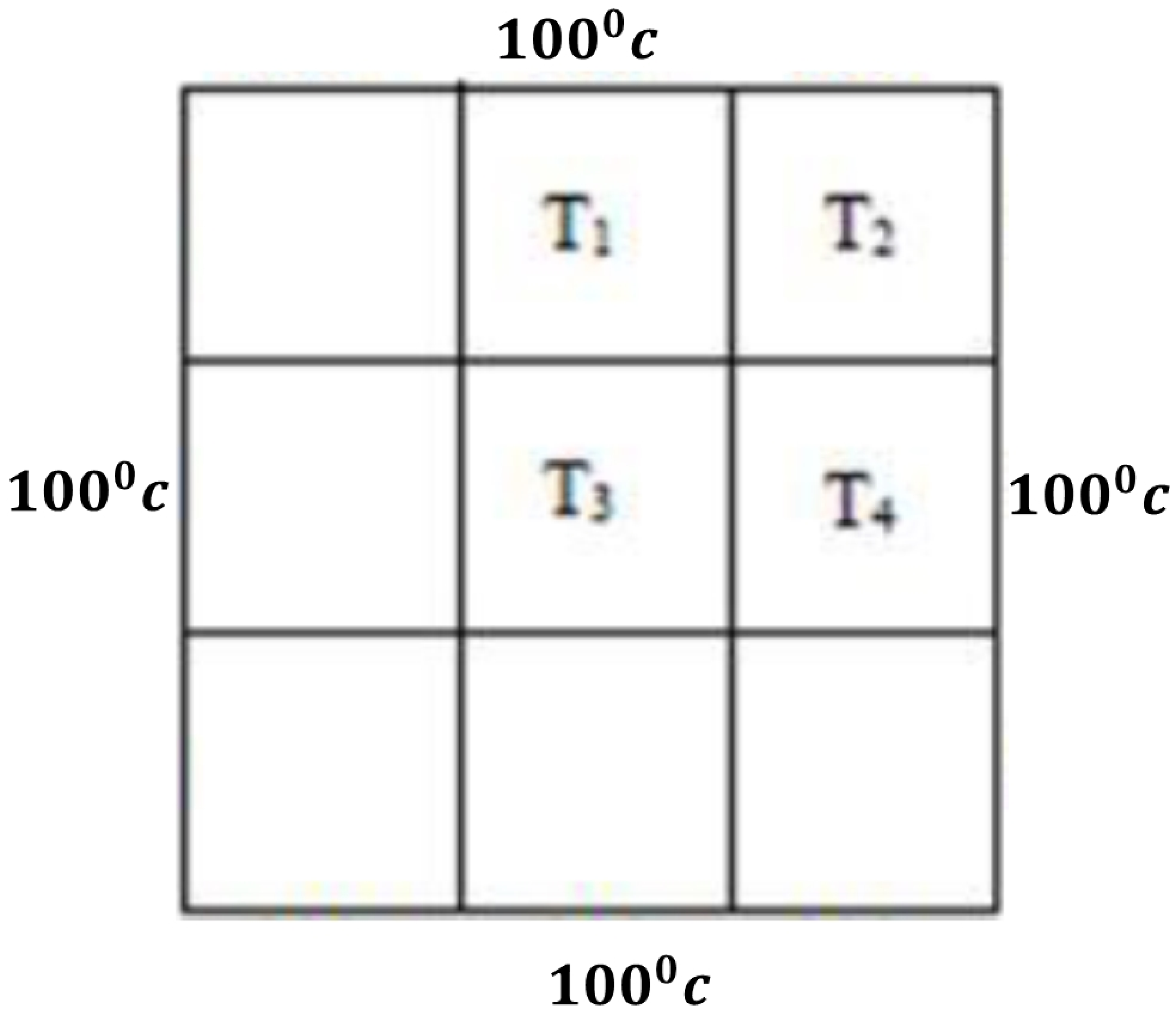
Solving equation (i), (ii), (iii) & (iv) we get;

$$T_1 = 75$$
$$T_2 = 50$$
$$T_3 = 50$$
$$T_4 = 25$$

3. The steady-state two dimensional heat flow in a metal plate is defined by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$. Consider a metal plate of size $30cm \times 30cm$, the boundaries of which are held at 100^0c. Calculate the temperature at interior points of the plate. Assume the grid size of $10cm \times 10cm$.

<u>Solution:</u>

Let T_1, T_2, T_3 & T_4 are the values of interior grid point.



Using the difference equation for the Laplace equation,

<u>At point T_1</u>

$$T_1 = \frac{1}{4}(100 + 100 + T_2 + T_3)$$

Or, $4T_1 - T_2 - T_3 = 200$ .... (i)

<u>At point T_2</u>

$$T_2 = \frac{1}{4}(100 + 100 + T_1 + T_4)$$

Or, $4T_2 - T_1 - T_4 = 200$ .... (ii)

At point T_3

$$T_3 = \frac{1}{4}(100 + 100 + T_1 + T_4)$$

Or, $4T_3 - T_1 - T_4 = 200$ ……………. (iii)

At point T_4

$$T_4 = \frac{1}{4}(100 + 100 + T_2 + T_3)$$

Or, $4T_4 - T_2 - T_3 = 200$ ……………. (iv)

Solving equation (i), (ii), (iii) & (iv) we get;

$T_1 = 100$

$T_2 = 100$

$T_3 = 100$

$T_4 = 100$

References:

- *E. Balagurusamy, Numerical Methods, Tata McGraw-Hill*

Please let me know if I missed anything or anything is incorrect.

Poudeljayanta99@gmail.com