DISCRETE STRUCTURES

Unit 1: Basic Discrete Structure.

1.1: **Sets**: Sets and subsets, Power set, Cartesian Product, Set Operations, Venn Diagrams, Inclusion-Exclusion Principle, Computer Representation of Sets.

- **Set**: A set is an unordered collection of objects.
- The objects in a set are called the elements or members of a set A. A set is said to contain its elements.
- ~~$a \in A$~~ $a \in A$ — ~~Does~~ a is an element of the set A.
  $a \notin A$ — a is not an element of the set A.
- Example 1: The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$
- Example 2: The set O of odd positive integers less than 10 can be expressed as $O = \{1, 3, 5, 7, 9\}$
- Example 3: A set can have unrelated elements. $\{a, 2, \text{Fred}, \text{New}\}$.
- Sometimes the brace notation is used to describe a set without listing all its members. Some members of the set are listed, and then ellipses (...) are used when the general pattern of the elements is obvious.
- Example 4: The set of positive integers less than 100 can be denoted by

$$\{1, 2, 3 \ldots 99\}$$

- Set builder notation: In set builder notation we characterize all those elements in the set by stating the property or properties they must have to be members. For instance, the set O of all odd positive integers less than 10 can be written as

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$

or, specifying the universe as the set of positive integers as.

$$O = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$$

We often use this type of notation to describe sets when it is impossible to list all the elements of the set. For instance, the set Q^+ of all positive rational numbers can be written as.

$$Q^+ = \{x \in R \mid x = p/q, \text{ for some positive integers } p \,\&\, q\}$$

$N = \{0, 1, 2, 3 \ldots\}$ the set of natural numbers

$Z = \{\ldots -2, -1, 0, 1, 2 \ldots\}$, the set of integers.

$Z^+ = \{1, 2, 3 \ldots\}$, the set of positive integers.

$Q = \{p/q \mid p \in Z;\ q \in Z;$ and $q \neq 0\}$, the set of rational numbers

R : the set of real numbers.

<u>Example 5</u>: The set $\{N, Z, Q, R\}$ is a set containing four elements, each of which is a set.

Definition 3: Two sets are equal if and only if they have the same elements. That is, if A and B are sets, then A and B are equal if and only if $\forall x (x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

Example 6: The sets $\{1, 3, 5\}$ and $\{5, 3, 1\}$ are equal, because they have the same elements. Order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of set is listed more than once, so $\{1, 3, 3, 3, 5, 5, 5\}$ is same as the set $\{1, 3, 5\}$ because they have same elements.

- Sets can be represented graphically using Venn diagrams. In Venn diagrams the universal set U, which contains all the objects under considerations, is represented by a rectangle. The universal set varies depending on which objects are of interest. Inside this rectangle, circles and other geometrical figures are used to represent sets. Sometimes points are used to represent the particular elements of sets. Venn diagrams are often used to indicate the relationships between sets.

U = universal set of all 26 ~~alphat~~ letter of English alphalbet

V: Set with all vowels of English alphabet.

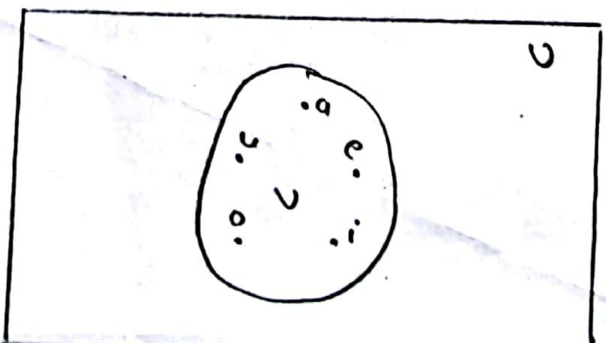

Figure: Venn Diagram for the set of vowels.

- The set which has no elements is called the empty set or null set and is denoted by $\emptyset$ or $\{\}$.

  Eg: The set of all positive integers that are greater than there squares is the null set.

- A set with one element is called a ~~sig~~ singleton set.

- A common error is to confuse the empty set with the set $\{\emptyset\}$, which is a singleton set. The single element of the set $\{\emptyset\}$ is the empty set itself.

<u>SUBSET</u>: A set A is said to be subset of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of set B.

We see that $A \subseteq B$ if and only if the quantification

$$\forall x \; (x \in A \rightarrow x \in B)$$

is true.

$\forall x \rightarrow$ For all x.

<u>Example</u>:

- The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10.
- The set of all people in China is a subset of the set of all people in China (that is, it is a subset of itself).
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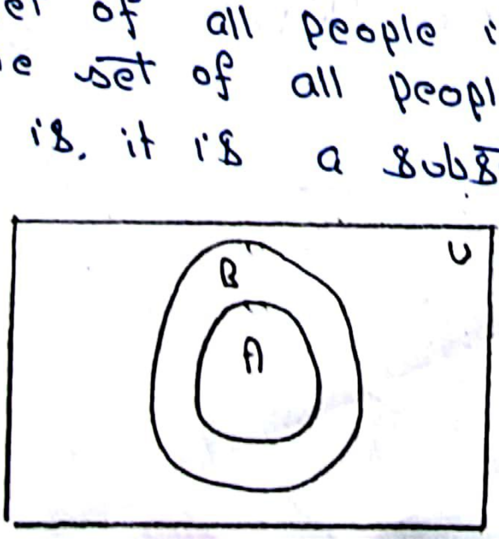

<u>Fig</u>: Venn diagram showing that A is a subset of B.

Theorem 1: For every set S,

(i) $\emptyset \subseteq S$ (ii) $S \subseteq S$

Theorem 1 shows that every non empty set S is ~~gua~~ guaranteed to have at least two subsets, the empty set and the set S itself, that is, $\emptyset \subseteq S$ and $S \subseteq S$.

- A is a proper subset of a set B, when A is subset of B but $A \neq B$.

$A \subset B \Rightarrow A$ is proper subset of B.

- For $A \subseteq B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element of x of B that is not an element of ~~B~~ A. That is, A is a proper subset of B if.

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A).$$

is true.

- One way to show that two sets have the same elements is to show that each set is subset of the other. We can show that if A and B are ~~subs~~ sets with $A \subseteq B$ and $B \subseteq A$, then $A = B$. ~~This turns out to be~~ That is $A = B$, where A and B are sets, if and only if $\forall x(x \in A \rightarrow x \in B)$ and $\forall x(x \in B \rightarrow x \in A)$ or equivalently if and only if $\forall x(x \in A \leftrightarrow x \in B)$.

Sets may have other sets as members. For instance, we have the sets.

$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $B = \{x \mid x$ is a subset of the set $\{a, b\}\}$.

- Here, $A = B$.
  Also $\{a\} \in A$ but $a \notin A$.

- Let S be a set. If there are exactly n distinct elements in S where n is non-negative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

- Example: Let A be the set of odd positive integers less than 10.
  $|A| = 5$.

- Example: Let S be the set of letters in the English alphabet. Then $|S| = 26$.

- Example: Because the null set has no elements, it follows that $|\emptyset| = 0$.

- A set is said to be infinite if it is not finite.
  eg. A set of positive integers.

- The Power Set: Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $P(S)$.

Example: The power set of the set $\{0, 1, 2\}$

$$P(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

- The empty set and the set itself are members of this set of subsets.

- What is the power set of the empty set?

→ The empty set has exactly one subset, namely, itself. Consequently, $P(\emptyset) = \{\emptyset\}$

- What is the power set of the set $\{\emptyset\}$

→ The set $\{\emptyset\}$ has exactly two subsets, namely, $\emptyset$ and the set $\{\emptyset\}$ itself. Therefore,

$$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

- If a set has n elements, then its power set has 2^n elements.

- The ordered <u>n-tuple</u> $(a_1, a_2 \ldots a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its nth element.

- Two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal. In other ~~order~~ words, $(a_1, a_2 \ldots, a_n) = (b_1, b_2, \cdots, b_n)$ if and only if $a_i = b_i$, for $i = 1, 2, \cdots, n$.

- 2-tuples are called ordered pairs. The ordered pairs (a,b) and (c,d) are equal if and only if $a=c$ and $b=d$.

- (a,b) and (b,a) are not equal unless $a=b$.

- Let A and B be sets. The Cartesian product of A and B, denoted by <u>$A \times B$</u> is the set of all ordered pairs (a,b), where <u>$a \in A$ and $b \in B$</u>. Hence,

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

- What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Solution: The Cartesian product $A \times B$ is

$$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$$

- The Cartesian products $A \times B$ and $B \times A$ are not equal, unless $A = \emptyset$ or $B = \emptyset$ (so that $A \times B = \emptyset$) or $A = B$.

- $A = \{1, 2\}$ $B = \{a, b, c\}$

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$

$B \times A = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

- The Cartesian product of the sets $A_1, A_2 \dots A_n$, denoted by $A_1 \times A_2 \times \dots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \dots, a_n)$, where a_i belongs to A_i, for $i = 1, 2, \dots, n$. In other words,

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}.$$

- What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$ $B = \{1, 2\}$ and $C = \{0, 1, 2\}$?

Solution: The cartesian product $A \times B \times C$ consists of all ordered ~~pairs~~ triples (a, b, c) where $a \in A$, $b \in B$ and $c \in C$. Hence,

$A \times B \times C = \{(0, 1, 0) \dots$

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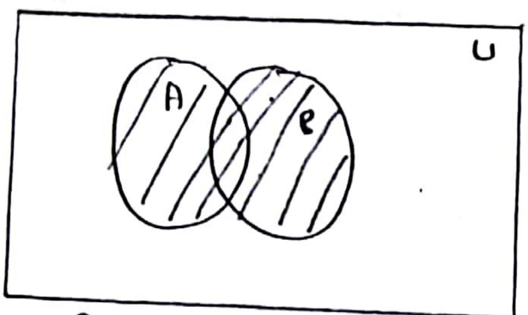
Set operations.

- Let A and B be sets. The union of sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

- An element x belongs to the union of the sets A and B if and only if x belongs to ~~x~~ A or x belongs to B. This tells us that

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$



Example: $A = \{1,3,5\}$
$B = \{1,2,3\}$

$A \cup B = \{1,2,3,5\}$

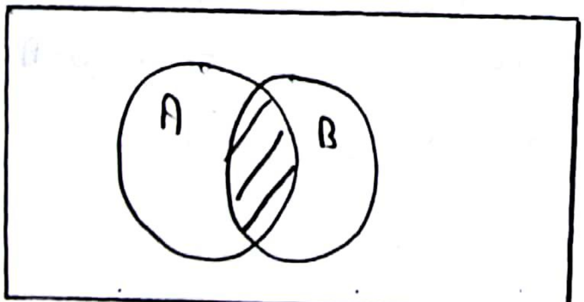
· $A \cup B$ is dashed

Fig: Venn Diagram representing $A \cup B$.

- Let A and B be sets. The intersection of sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

- An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B.

$$\therefore A \cap B = \{x \mid x \in A \wedge x \in B\}$$



Example: $A = \{1,3,5\}$
$B = \{1,2,3\}$

$A \cap B = \{1,3\}$

Fig. Venn Diagram representing $A \cap B$

- Two sets are called disjoint if their intersection is the empty set.

Example: Let $A = \{1, 3, 5, 7, 9\}$ $B = \{2, 4, 6, 8, 10\}$.
Because $A \cap B = \emptyset$, A and B are disjoint.

- Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

- An element x belongs to the difference of A and B if and only if $x \in A$ and $x \notin B$.

$\therefore$ $A - B = \{x \mid x \in A \wedge x \notin B\}$

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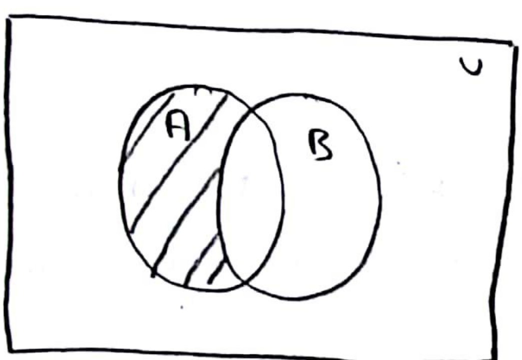

A-B is

Fig: Venn diagram for difference of A and B.

Example: $A = \{1, 3, 5\}$ $B = \{1, 2, 3\}$

$A - B = \{5\}$

$B - A = \{2\}$

- Let U be the universal set. The complement of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. In other words, the complement of set A is $U - A$.
- An element belongs to $\overline{A}$ if and only if $x \notin A$.

$\overline{A} = \{x \mid x \notin A\}$.

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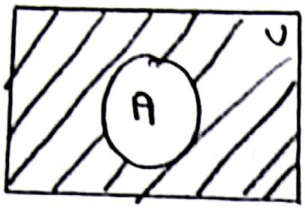


Fig: Venn Diagram for the complement of the set A.

Example:

$A = \{a, e, i, o, u\}$.

$U = \{x \mid x$ is the set of letters of the English alphabet $\}$.

$\overline{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$

- **Set Identities:**

Identity	Name.
• $A \cup \varnothing = A$ • $A \cap U = A$	Identity laws
• $A \cup U = U$ • $A \cap \varnothing = \varnothing$	Domination Laws
• $A \cup A = A$ • $A \cap A = A$	~~Id~~ Idempotent laws
• $(\overline{A}) = A$	Complementation laws
• $A \cup B = B \cup A$ • $A \cap B = B \cap A$	Commutative Laws
• $A \cup (B \cup C) = (A \cup B) \cup C$ • $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
• $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ • $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws

• $\overline{A \cup B} = \overline{A} \cap \overline{B}$ • $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's Laws
• $A \cup (A \cap B) = A$. • $A \cap (A \cup B) = A$.	Absorption Laws
• $A \cup \overline{A} = U$ • $A \cap \overline{A} = \emptyset$	Complement Laws.

• The union of a collection of sets is the set that contains those elements that are members at least one set in the collection.

We use the notation,

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

to denote the union of the sets $A_1, A_2 \cdots A_n$.

• The intersection of a collection of sets is the set that contains those elements that are members of all sets in the collection.

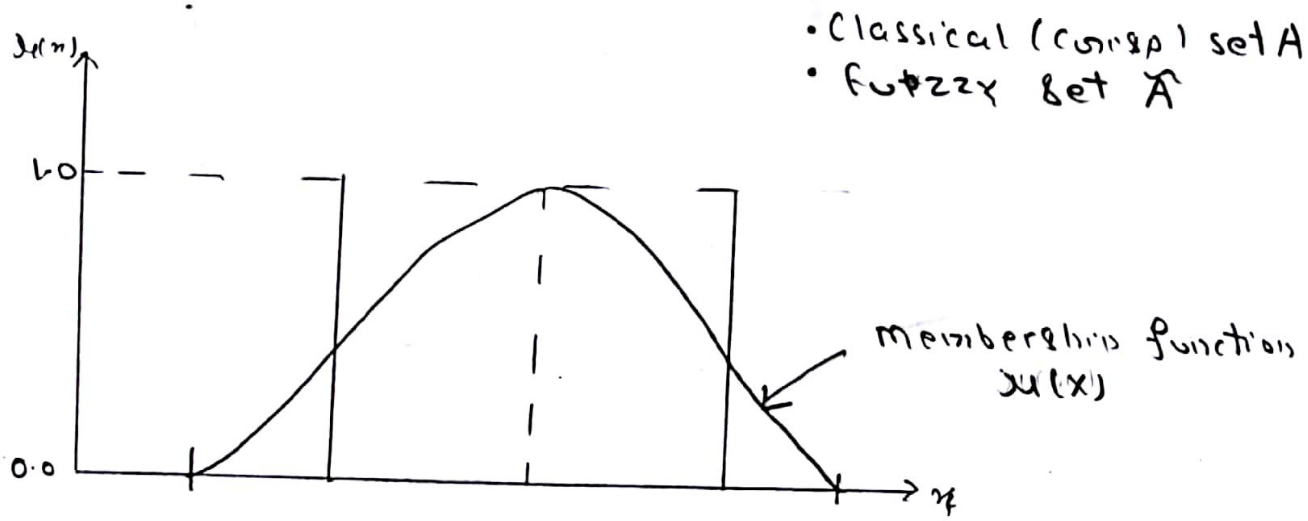
We use the notation

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

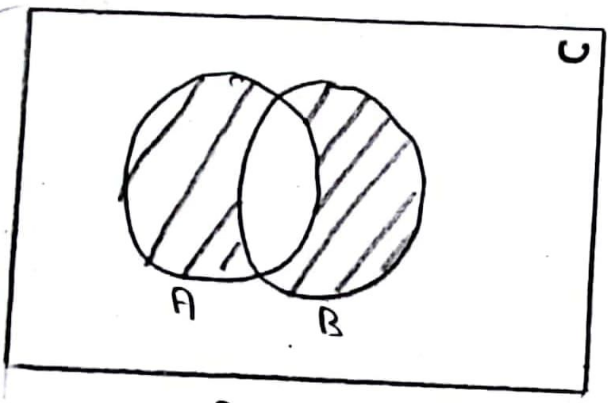
to denote intersection of the sets $A_1, A_2, \cdots, A_n$.

Fuzzy Sets

Definition: Let X be a space of points, with a generic element of X denoted by x. Thus $X = \{x\}$.
A fuzzy set A in X is characterized by a membership function $f_A(x)$ which associates with each point in X a real number in the interval $[0,1]$, with the values of $f_A(x)$ at x representing the "grade of membership" of x in A. Thus, the nearer the value of $f_A(x)$ to unity, the higher the grade of membership of x in A.



- The symmetric difference of two sets A and B is the set $(A-B) \cup (B-A)$ and is denoted by $A \Delta B$. The shaded part of the given Venn diagram represents $A \Delta B$. $A \Delta B$ is the set of all those elements which belongs to either A or B but not to ~~B~~ both.



$A \Delta B$

Inclusion and Exclusion Principle.

Let A and B be any finite sets.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

To find the number $n(A \cup B)$ of elements in the union of A and B, we add $n(A)$ and $n(B)$ and then we subtract $n(A \cap B)$; that is we "include" $n(A)$ and $n(B)$ and we "exclude" $n(A \cap B)$. This follows from the fact that, when we add $n(A)$ and $n(B)$, we have counted the elements of $n(A \cap B)$ twice.

- For any finite sets, A, B, C we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$$

That is we "include" $n(A), n(B), n(C)$, we "exclude" $n(A \cap B)$, $n(A \cap B)$, $n(A \cap C)$, and finally "include" $n(A \cap B \cap C)$.

Example: Find the number of students at a college taking ~~at~~ at least one of the languages French, German and Russian, given the following data.

→ 65 study French, 20 both French & Germ.
45 Study German, 25 French & Russian, 8 study all three.
42 Study Russian, 15 German & Russian

By Inclusion-Exclusion Principle.

$$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$$

$$= 100.$$

- Now, suppose we have finite number of sets, say, $A_1, A_2, A, \ldots, A_m$. Let S_k be the sum of the cardinalities.

$$n(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})$$

of all possible k-tuple intersections of given m sets. Then we have the following general Inclusion-Exclusion principle.

$$n(A_1 \cup A_2 \cup A_3 \cdots \cup A_m) = S_1 - S_2 + S_3 - \cdots + (-1)^{m-1} S_m.$$

- **Computer Representation of Sets:**

Assume that the universal set U is finite (and of reasonable size). First, specify an arbitary ordering of the elements of U, for instance $a_1, a_2, \cdots, a_n$. Represent a subset A of U with the bit string of length n, where ith bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

- **Example** Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit string represents the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Solution: The bit string that represents the set of odd integers in U, namely, $\{1, 3, 5, 7, 9\}$ is 10 1010 1010.

- The subset of all even integers in U, namely $\{2, 4, 6, 8, 10\}$ is represented by.

  01 01 01 01 01.

- The set of all integers in U that do not exceed 5, namely, $\{1, 2, 3, 4, 5\}$, is represented by the string.

  11111 00000.

Example:

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

A: Odd $= \{1, 3, 5, 7, 9\}$

Bit string for A is 10 1010 1010

Bit string for the $\overline{A}$ = 01 0101 0101

$\overline{A}$ is obtained by replacing 0s with 1s and vice versa.

Example:

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 2, 3, 4, 5\}$ 11111 00000

$B = \{1, 3, 5, 7, 9\}$ 10 1010 1010

- $A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$.

Bit string to represent $A \cup B$ = 11 11100 000 $\vee$ 10 1010 1010

= 11 1110 1010

- Bit string to represent $A \cap B$ = 11 1110 0000 $\wedge$ 10 10101010

= 10 1010 0000

$A \cap B = \{1, 3, 5\}$

- The bit string for the union is the bitwise OR of the bit strings for the two sets.
- The bit string for the intersection is the bitwise AND of the bit strings for the two sets.

FUNCTIONS

Definition: Let A and B be nonempty sets. A function from A to B is an assignment of exactly one element of B to each element of A. We write $f(a)=b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f: A \to B$.

- Functions are also called mapping and transformations.

- If f is a function from A to B, we say that A is the domain of f and B is codomain of of f. If $f(a)=b$, we say that b is the image of a and a is a preimage of b. The ranges of f is the set of all images of elements of A. Also, if ~~is a~~ f is a function from A to B, we say that f maps A to B.

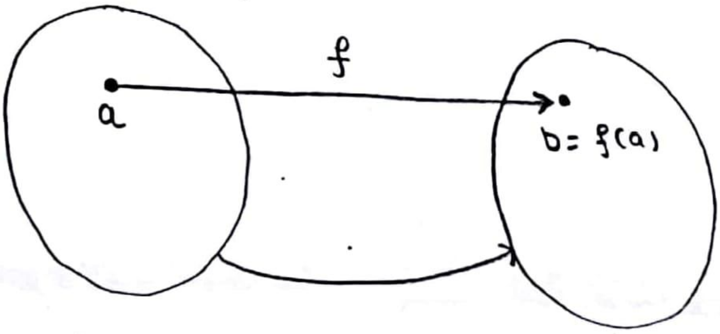


Fig: The function f maps A to B

- Two functions are equal when they have the same domain, have the same codomain, and ~~map~~ map elements of their common domain to the same elements in their common codomain. If we change either the domain or the codomain of a function, then we obtain a different function. If we change mapping of elements, then we also obtain a different function.

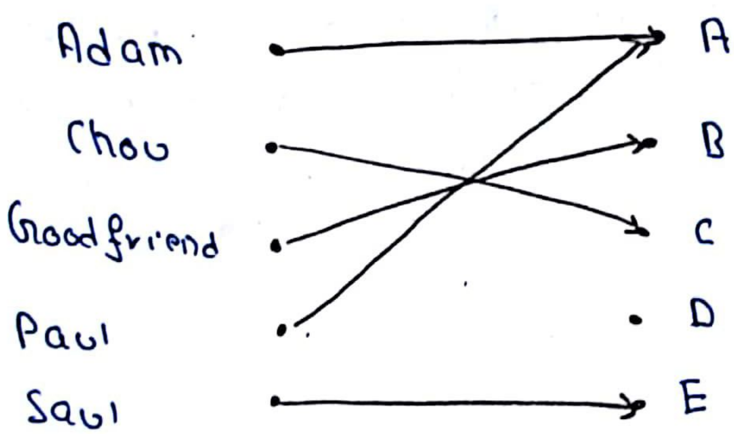
<u>Examples</u>: Students = { Adam, Chou, Goodfriend, Paul, Saul }

Grades = { A, B, C, D, E }

Let G be the ~~the~~ function that assigns a grade to a student.

G(Adams) = A.

- The domain of G is the set { Adam, Chou, Goodfriend, Paul, Saul } and the codomain is the set {A, B, C, D, E}
The range of G is the set { A, B, C, E }, because each grade except D is assigned to students.



<u>Example</u>: Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, f(11010) = 10. Then, the domain of f is the set of all bit strings of length 2 or greater, and both the codomain and the range are the set {00, 01, 10, 11}.

<u>Example</u>: Let $f: Z \to Z$ assign the square of an integer. Then, $f(x) = x^2$, where the domain of f is the set of all integers, we take the codomain of f to be the set of all integers, and the range of f is

the set of all integers that are perfect squares, namely $\{0, 1, 4, 9 \ldots\}$

<u>Example</u>. The domain and codomain of functions are often ~~spef~~ specified in programming languages. For instance, the java statement.

int floor (float real) {...}

and the pascal statement.

function floor (x: real): integer.

both state that domain of the floor function is the set of real numbers and its codomain is the set of integers.

- Let f_1 and f_2 be functions from A to R. Then $f_1 + f_2$ and $f_1 \cdot f_2$ are also functions from A to R defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x) \cdot f_2(x).$$

- <u>Example</u>: Let f_1 and f_2 be the functions from R to R such that $f_1(x) = x^2$ and $f(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

<u>Solution</u>: From the definition of the sum and product of functions, it follows that.

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x.$$

$$(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) = x^2 (x - x^2) = x^3 - x^4.$$

- Let f be the function from the set A to the set B and let S be a subset of A. The image of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so.

$$f(S) = \{ t \mid \exists s \in S \, (t = f(s)) \}.$$

We use the shorthand $\{ f(s) \mid s \in S \}$ to denote this set.

- Here, $f(S)$ denotes a set, and not the value of the function f for the set S.

Example. Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$

- Injective function: A function f is said to be one-to-one, or injective, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an injection if it is one-to-one.

what is difference between Relation and function?

~~If~~ The function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$.

- We can express that f is one-to-one using quantifiers as $\forall a \forall b \, (f(a) = f(b) \rightarrow a = b)$ or equivalently $\forall a \forall b \, (a \neq b \rightarrow f(a) \neq f(b))$, where the universe of ~~disclo~~ discourse is the domain of the function.

Example. Show $f(x) = 3x - 2$ is injective.

$$f(x_1) = f(x_2)$$

$$3x_1 - 2 = 3x_2 - 2$$

$$x_1 = x_2.$$

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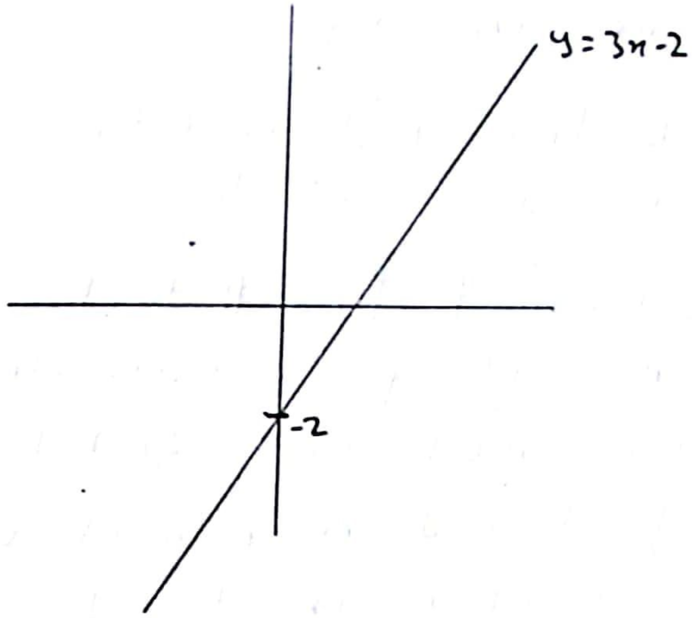
$x = 3$

$y = f(x) = 3 \times 3 - 2 = 7$

No other value of x will give $y = 3$.



- Is $f(x) = x^2$, injective.

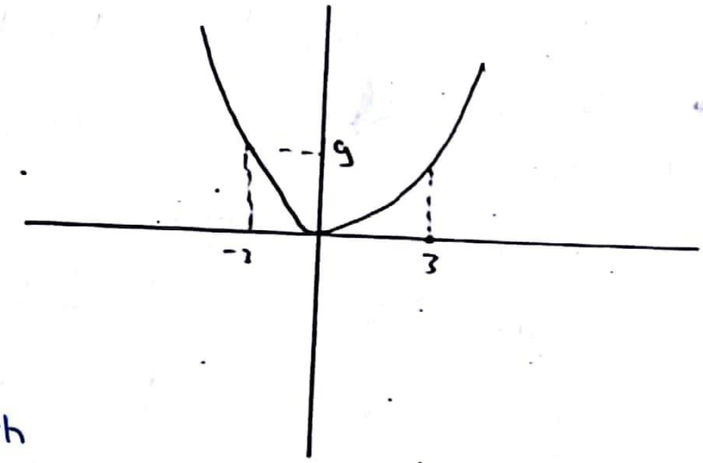
$f(x_1) = f(x_2)$

$x_1^2 = x_2^2$

$\pm x_1 = \pm x_2$

Let, $x_1 = 3$

$+3 = -3$ ✗



- we get $y = 9$ for both

$f(3) = 9$

$f(-3) = 9$

therefore, $f(x) = x^2$ is not injective function.

- A function f whose domain and codomain are subsets of the set of real numbers is called increasing if $f(x) \le f(y)$, and strictly increasing if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called decreasing if $f(x) \ge f(y)$, and strictly decreasing if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f.
- A function f is increasing if $\forall x \forall y (x < y \to f(x) \le f(y))$, strictly increasing if $\forall x \forall y (x < y \to f(x) < f(y))$, decreasing if $\forall x \forall y (x < y \to f(x) > f(y))$ and strictly decreasing if $\forall x \forall y (x < y \to f(x) > f(y))$ where the universe

of discourse of f.

- A function that is either strictly increasing or strictly decreasing must be one-to-one.

- A function f from A to B is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
- A function f is onto if $\forall y \exists x (f(x) = y)$, where the domain of x is the domain of the function and the domain for y is the codomain of the function.
- In case of surjective function, Codomain = Range.
- Example. Is $f(x) = 5x + 2$ is surjective for $\forall x \in R$?

$f: R \to R$.

$y = f(x)$

$y = 5x+2$

$x = \frac{y-2}{5}$.

~~x=+~~ $y = 1$

$x = \frac{1-2}{5} = -1/5 \rightarrow$ it is real no.

for $y = \cancel{8}\, 7$

$x = \frac{7\cancel{8}-2}{5} = \cancel{B}1 \rightarrow$ it is real No.

So, $f(x) = 5x + 2$ is surjective when domain and Co-domain are Real numbers, i.e for every 'y' there is atleast one 'x'.

- Is $f(x) = 5x+2$ is surjective for $\forall x \in Z$?

$f: Z \to Z$

$x = \frac{y-2}{5}$

for $y = 7$

$x = 5$

for $y = 5$

$x = 3/5 \rightarrow$ this is not an integer.

$\therefore$ $f(x) = 5x + 2$ is not a surjective function when the domain is a real number.

- **Example:** If we restrict the function $f(x) = x^2$ to a function of all non negative real numbers, then f is invertible.

- **Composite functions.**

Let g be the function from set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted by $f \circ g$, is defined by.

$$(f \circ g)(a) = f(g(a))$$

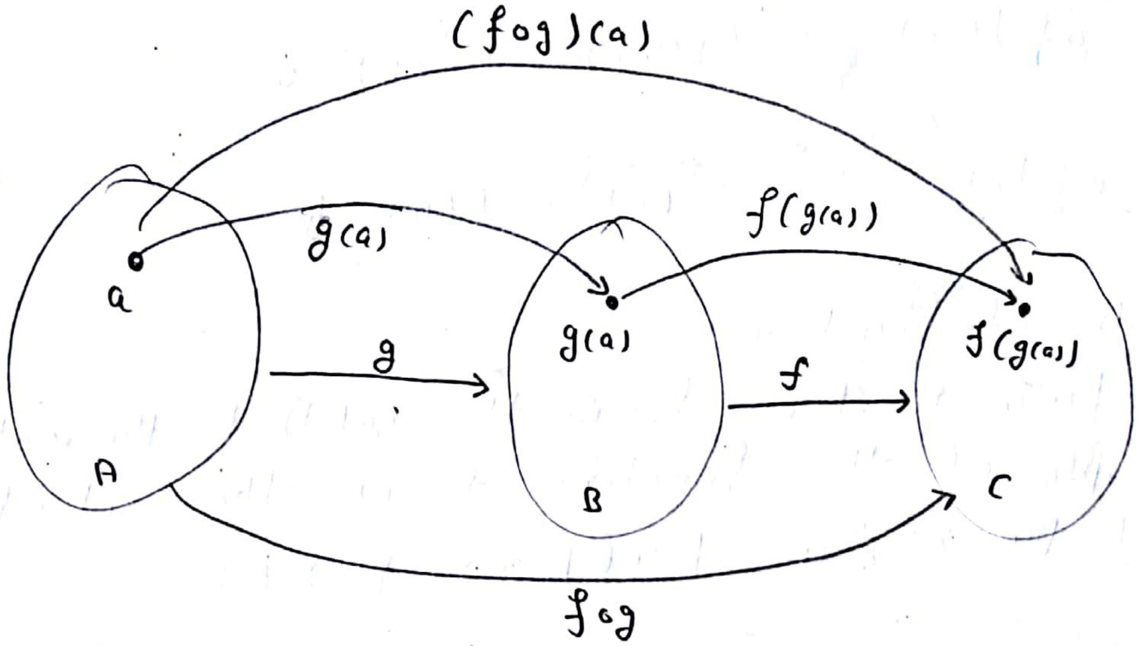


Fig: The composition of functions 'f' and 'g'

Example: $f(x) = 2x+3.$
$g(x) = 3x+2.$

The composition $f \circ g$ is

$(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)+3 = 6x+7$
$= \cancel{6x+6+2 = 6x}$

2

$(g \circ f)(x) = g(f(x)) = g(2x+3) = 3(2x+3)+2 = 6x+11$

- $f \circ g$ and $g \circ f$ are not equal that is commutative ~~ho~~ law does not hold for composition of functions.
- When the composition of a function and its inverse is formed, in either order, an identity function is ~~formed~~ obtained. Let f is a bijective function from set A to set B. Then the inverse function f^{-1} exists and is a one-to-one correspondence from B to A. The inverse function reverses the correspondence of the original function, so $f^{-1}(b) = a$, when $f(a) = b$, and $f(a) = b$ when $f^{-1}(b) = a$. Hence,

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a.$$

and

$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b.$$

The graphs of function.

- Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$

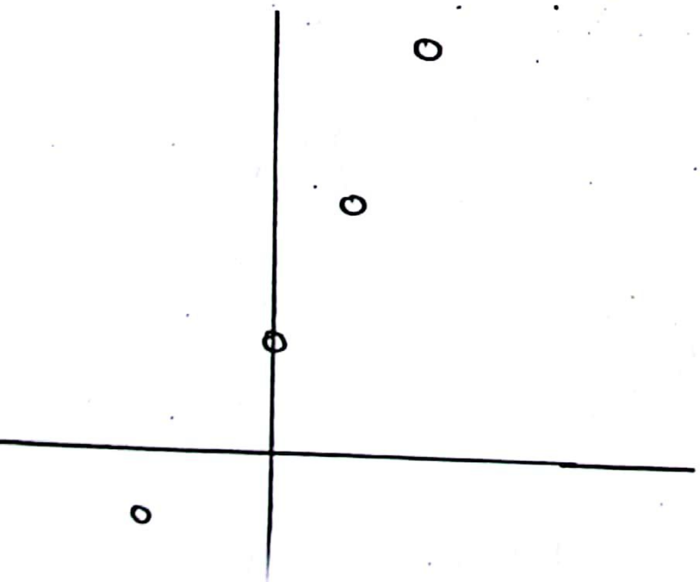


Fig: The graph of $f(n) = 2n+1$ from Z to Z

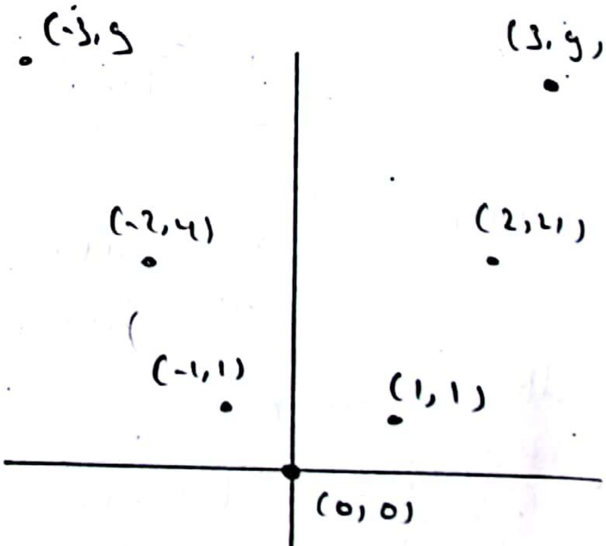


Fig: The graph of $f(x) = x^2$ from Z to Z

~~fee~~

Ceiling Function: The ~~ceil~~ ceiling function assigns to the real number x the smallest integer that is greater than equal to x. The value of ceiling function at x is denoted by $\lceil x \rceil$.

Floor function: The floor function assigns to the real number x the largest that is less than or equal to x. The value of floor function at x is denoted by $\lfloor x \rfloor$.

Example:

Ceiling	Floor
$\lceil 3.1 \rceil = 4$	$\lfloor 3.1 \rfloor = 3$
$\lceil 7 \rceil = 7$	$\lfloor 7 \rfloor = 7$
$\lceil 6.9 \rceil = 7$	$\lfloor 6.9 \rfloor = 6$

Boolean Function:

- A function whose arguments, as well as, the function itself, assume values of from a two-element set usually $\{0, 1\}$.
- A Boolean function is described by an algebraic expression called Boolean Expression which consists of binary variables, the constants 0 and 1, and the logic operation symbols.

Example: $F(A,B,C,D) = A + B\bar{C} + D$.

Exponential function: If b is any number such that $b>0$ and $b \neq 1$ then an exponential function is a function in a form,

$$f(x) = b^x$$

where, b is called base and x can be any real number. eg. $f(x) = 2^x$, $f(x) = e^{-x^2}$

• Fuzzy Set

If X is an universe of discourse and x is a particular element of X, then a fuzzy set A defined on X and can be written as a collection of ordered pairs.

$$A = \{ (x, \mu_{\tilde{A}}(x)), x \in X \}$$

$\mu_{\tilde{A}}(x)$ = Degree of membership of x in A.

• Example: Let $X = \{a, b, c, d, e\}$ be the reference set of students.
Let $\tilde{A}$ be the fuzzy set of "smart" students, where smart is fuzzy term.

$$\tilde{A} = \{ (a, 0.4) (b, 0.5) (c, 1) (d, 0.9) (e, 0.8) \}$$

Here, $\tilde{A}$ indicates smartness that smartness of b is 0.5.

Membership function

- Membership functions ~~were first introduced~~ characterize fuzziness (i.e all the information in fuzzy set), whether the elements in fuzzy sets are discrete or continous.
- Membership functions are represented by graphical forms.

- A fuzzy set $\tilde{A}$ in the universe of information U can be defined as a set of ordered pairs and it can be represented mathematically as.

$$\tilde{A} = \{ (y, \mu_{\tilde{A}}(y)) \mid y \in U \}$$

Here, $\mu_{\tilde{A}}(\cdot)$ = membership function of $\tilde{A}$; this assumes ~~that~~ values in the range from 0 to 1, i.e $\mu_{\tilde{A}}(\cdot) \in [0,1]$. The membership function $\mu_{\tilde{A}}(\cdot)$ maps U to membership space M.

The dot $(\cdot)$ in the membership function described above, represents the element in a fuzzy set; whether it is discrete or continuous.

- Features of membership function.

Core: For a fuzzy set $\tilde{A}$, the core of a membership function is that region of universe that is characterize by full membership in the set. Hence, core consists of all those elements y of the universe of information such that,

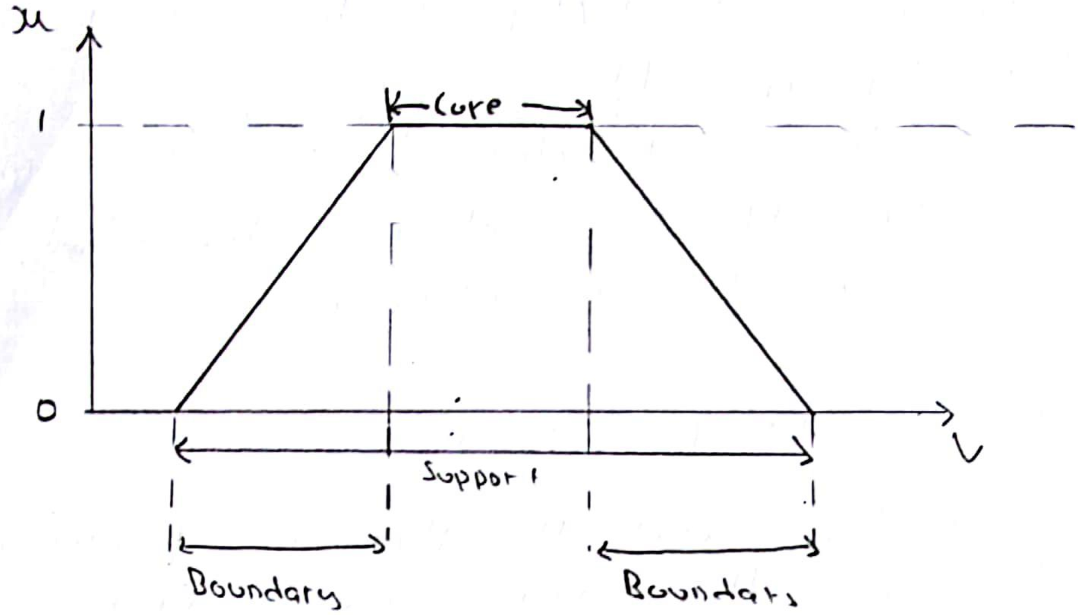
$$\mu_{\tilde{A}}(y) = 1$$

Support: For any fuzzy set $\tilde{A}$, the support of membership function in the region of universe that is characterize by a non-zero membership ind the set. Hence core consists of all those elements y of the universe of information such that,

$$\mu_{\tilde{A}}(y) > 0$$

Boundary: For a fuzzy set $\tilde{A}$, the boundary of a membership function is the region of the universe that is characterized by a non zero but incomplete membership in the set. Hence, core consists of all those elements y of the universe of information such that,

$$1 > \mu_{\tilde{A}}(y) > 0$$

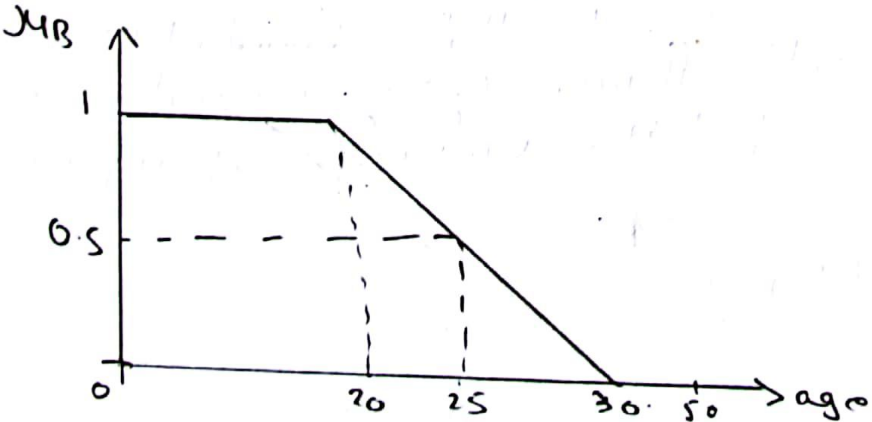


Features of membership function.

Example: A fuzzy ~~set~~ subset YOUNG is defined, which answers the question "to what degree is a person x young?" To each person in universe of discourse (set of people), we have to assign a degree of membership in the fuzzy subset YOUNG. The easiest way to do this is with a membership function based on the person's age.

young(x) = { 1, if age(x) <= 20
(30 - age(x))/10, if 20 < age(x) <= 30
0, if age(x) > 30 }

A graph looks like.



Person	Age	degree of Youth
Johan	10	1.0
Edwin	21	0.90
Chinwai	28	0.2
Raj	83	0

- Representation of fuzzy set.

$$\tilde{A} = \left\{ \frac{\mu_{\tilde{A}}(y_1)}{y_1} + \frac{\mu_{\tilde{A}}(y_2)}{y_2} + \frac{\mu_{\tilde{A}}(y_3)}{y_3} + \dots \right\}$$

$$= \left\{ \sum_{i=1}^{n} \frac{\mu_{\tilde{A}}(y_i)}{y_i} \right\}$$

Example:

$$\tilde{A} = \left\{ \frac{1.0}{\text{Johan}} + \frac{0.90}{\text{Edwin}} + \frac{0.20}{\text{Chin wai}} + \frac{0}{\text{Raj}} \right\}$$

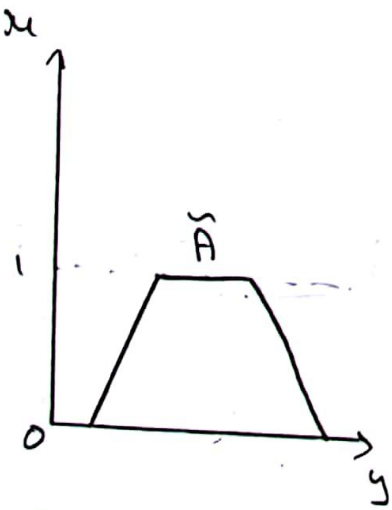
Operations on Fuzzy Sets.

- Having two fuzzy sets $\tilde{A}$ and $\tilde{B}$, the universe of information U and an element y of the universe, the following relations express the union, intersection and complement operation on fuzzy sets.

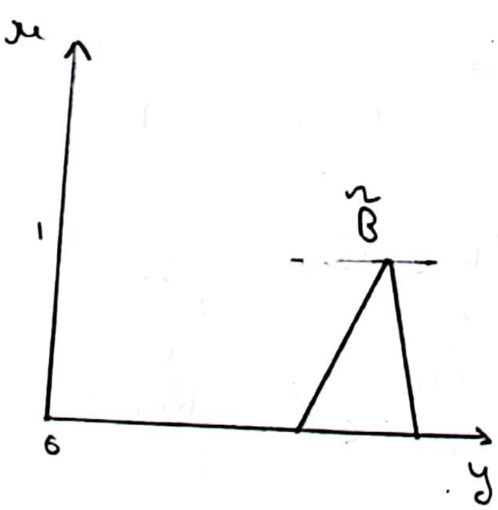
Union/ fuzzy 'OR'

$$\mu_{\tilde{A} \cup \tilde{B}}(y) = \mu_{\tilde{A}} \vee \mu_{\tilde{B}} \quad \forall y \in U$$

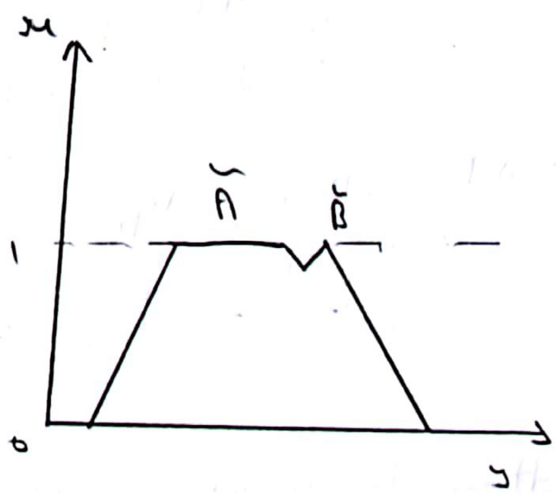
Here $\vee$ represents the 'max' operation.



Fuzzy set $\tilde{A}$



Fuzzy set $\tilde{B}$



Union of $\tilde{A}$ & $\tilde{B}$

- **Intersection / Fuzzy 'AND'**

$$\mu_{\tilde{A} \cap \tilde{B}}(y) = \mu_{\tilde{A}} \wedge \mu_{\tilde{A}\tilde{B}} \quad \forall y \in U$$

Here $\wedge$ represents the 'min' operation.

- **Complement / Fuzzy 'NOT'**

$$\mu_{\bar{A}} = 1 - \mu_{\tilde{A}}(y) \quad y \in U$$

Example:

$$\tilde{A} = \left\{ \frac{1.0}{\text{Johan}} + \frac{0.90}{\text{Edwin}} + \frac{0.20}{\text{Chinwai}} + \frac{0}{\text{Raj}} \right\}$$

$$\tilde{B} = \left\{ \frac{1.0}{\text{Johan}} + \frac{0.8}{\text{Edwin}} + \frac{0.1}{\text{Chinwai}} + \frac{0.1}{\text{Raj}} \right\}$$

$$\tilde{A} \cup \tilde{B} = \max\{ \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x) \}$$

$$= \left\{ \frac{1.0}{\text{Johan}} + \frac{0.9}{\text{Edwin}} + \frac{0.2}{\text{Chinwai}} + \frac{0.1}{\text{Raj}} \right\}$$

$$\tilde{A} \wedge \tilde{B} = \min\{ \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x) \}$$

$$= \left\{ \frac{1.0}{\text{Johan}} + \frac{0.8}{\text{Edwin}} + \frac{0.1}{\text{Chinwai}} + \frac{0}{\text{Raj}} \right\}$$

$$\bar{\tilde{A}} = 1 - \mu_A(x) = \left\{ \frac{0}{\text{Johan}} + \frac{0.1}{\text{Edwin}} + \frac{0.8}{\text{Chinwai}} + \frac{1}{\text{Raj}} \right\}$$

- A sequence is a function from a subset of the set of integers (usually either the set $\{0, 1, 2 \ldots\}$ or the set $\{1, 2, 3 \ldots\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

Example: Consider the sequence, $\{a_n\}$, where.

$$a_n = \frac{1}{n}$$

The list of the terms of this sequence, begining with a_1, namely,

$$a_1, a_2, a_3, a_4, \ldots$$

starts with

$$1, 1/2, 1/3, 1/4, \ldots$$

- A geometric ~~sequence~~ progression is the sequence of the form

$$a, ar, ar^2, \ldots, ar^n$$

where the initial term 'a' and the common ratio 'r' are real numbers.

- A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$

- Example

The sequences $\{b_n\}$ with $b_n = (-1)^n$

$\{c_n\}$ with $c_n = 2 \cdot 5^n$

$\{d_n\}$ with $d_n = 6 \cdot (1/3)^n$

are geometric progression.

Initial term and common ratio for $\{b_n\}$ are 1 & -1,

" " " " " " $\{c_n\}$ are 2 & 5

" " " " " " $\{d_n\}$ are 6 & $1/3$

If we start with $n=0$.
The list of terms $b_0, b_1, b_2, b_3, \ldots$ begins with
$1, -1, 1, -1, \ldots$

- The list of terms $c_0, c_1, c_2, c_3, \ldots$ begins with
$2, 10, 250, 1250, \ldots$

- The list of terms $d_0, d_1, d_2, d_3 \ldots$ begins with
$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$

- An arithmetic progression ~~begins~~ is a sequence of the form
$a, a+d, a+2d, \ldots, a+nd$
where the initial term 'a' and the common difference 'd' are real numbers.

- An arithmetic progression is a discrete analogue of the linear function $f(x) = dx+a$.

- <u>Example</u>
The sequence $\{S_n\}$ with $S_n = -1+4n$ is arithmetic progression with initial term $= -1$
common difference $= 4$

If we start with $n=0$, the list of terms $S_0, S_1, S_2, S_3, \ldots$ begins with
$-1, 3, 7, 11$

- The sequence $\{t_n\}$ with $t_n = 7-3n$
initial term $= 7$, common difference $= -3$
$\{t_0, t_1, t_2, t_3, \ldots\} = \{7, 4, 1, -2, \ldots\}$

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- **Example**: How can we produce the terms of a sequence if the first 10 terms are: 5, 11, 17, 23, 29, 35, 41, 47, 53, 59?

- Summations:

Following notation is used to describe the sum of the terms.

$a_m, a_{m+1}, \dots, a_n$

from the sequence $\{a_n\}$. We use the notation.

$$\sum_{j=m}^{n} a_j, \quad \sum\nolimits_{j=m}^{n} a_j, \quad \text{or} \quad \sum\nolimits_{m \le j \le n} a_j$$

to represent

$a_m + a_{m+1} + \dots + a_n$.

Here, the variable j is called the index of summation and the choice of the letter 'j' as the variable is arbitrary, that is, we could have used any letter, such as i or k. Or, in notation.

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k.$$

- Here, the index of summation runs through all integers starting with its lower limit m and ending with its ~~lower limit~~ upper limit n. A large ~~upper~~ upper case Greek letter Sigma, Σ, is used to denote summation.
- The usual laws for arithmetic apply to summations. For example, when a and b are real numbers, we have $\sum_{j=1}^{n} (ax_j + by_j) = a\sum_{j=1}^{n} x_j + b\sum_{j=1}^{n} y_j$, where $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ are real numbers.

Example Express the sum of the first 100 terms of the sequence $\{a_n\}$, where $a_n = \frac{1}{n}$, for $n = 1, 2, 3 \ldots$

Solution: lower limit = 1

upper limit = 100

$$\sum_{j=1}^{100} \frac{1}{j}$$

Example: What is the value of $\sum_{k=4}^{8} (-1)^k$?

Example: Double summations arise in many contexts (as in the analysis of nested loops in computer programs)

An example of double summations is

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij$$