- Combinatorics: The study of arrangements of objects.
- Counting is required to determine whether there are enough telephone numbers or Internet protocol addresses to meet demand. Counting techniques are used extensively when probabilities of events are computed.
- Basic counting principles
  - ↳ The product rule
  - ↳ The sum rule.

THE PRODUCT RULE: Suppose that a procedure can be broken down into ~~a~~ sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Example 1 A new company with just two employees, Sanchez and patel, rents a floor of a building with 12 offices. How ~~may~~ many ways are there to assign different offices to these two employees?

Solution:

$$12 \cdot 11 = 132$$

Example 2: The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution:

$$26 \cdot 100 = 2600$$

- Extended product rule.

Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \ldots, T_m$ in sequence. If each task T_i, $i = 1, 2, \ldots, n$, can be done in n_i ways, regardless of how previous tasks were done, then there are $n_1 \cdot n_2 \cdots n_m$ ways to carry out the procedure.

- Example: How many different bit strings of length seven are there?

Solution: Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1.

Therefore, the product rule shows that there are a total of $2^7 = 128$ different bit strings of length seven.

Example How many different license plates are available if each plate contains a sequence of three letters followed by three digits?

Solution:

$$\underbrace{\_\ \_\ \_}_{\text{26 choices for each letter}}\quad \underbrace{\_\ \_\ \_}_{\text{10 choices for each digit}} \qquad 26\cdot 26\cdot 26\cdot 10\cdot 10\cdot 10$$

Example: Counting functions.

How many functions are there from a set with m elements to a set with n elements?

Solution: A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain.

Hence, by product rule there are $n\cdot n \cdots n = n^m$ functions from a set with m elements to one with n elements.

For example, there are 5^3 different functions from a set with three different elements to a set with 5 elements.

Example 7 Counting one-to-one functions
How many one to one functions are there from a set with m elements to set with n elements?

Solution: When $m \geq n$ there are no one-to-one functions from a set with m elements to a set with n elements.
Now let $m \leq n$. ~~Suppose~~

$n(n-1)(n-2) \cdots (n-m+1)$

- for example, there are $5 \cdot 4 \cdot 3 = 60$ one-to-one functions from a set with three elements to a set with five elements.

The Sum Rule: If a task can be ~~cons~~ done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways in the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the ~~tak~~ task.

Extentended Sum Rule:
~~We can~~ Suppose that a task can be done in one of n_1 ways, in one of n_2 ways, or in one of n_m ways, where none of the set of n_i ways of doing the task is same as any of the set of n_j ways, for all pairs i and j with $1 \leq i \leq j \leq m$. Then number of ways to do the

task is $n_1 + n_2 + \dots + n_m$.

Example: A student can chose a computer project from one of three lists. The three lists contain 23, 15 and 19 possible projects, respectively. No project is on more than one list. How many projects are there to choose from?

Solution:

$$23 + 15 + 19 = 57.$$

Example: Each user on a computer system has a password, which is six to eight characters long, where each character is an upper case or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution

$$P = P_6 + P_7 + P_8$$

$$P_6 = 36^6 - 26^6$$

$$P_7 = 36^7 - 26^7$$

$$P_8 = 36^8 - 26^8$$

THE PIGEONHOLE PRINCIPLE

It states that if there are more pigeons than pigeonholes, then there must be atleast ~~two~~ one pigeon hole, with atleast two pigeons in it.

Theorem 1: The Pigeonhole Principle

If k is a positive integer and k+1 or more objects are placed into k boxes, then there is atleast one box containing two or more of the objects.

Proof: Proof by

Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction, because there are atleast k+1 objects.

- The Pigeonhole Principle is also called the Dirichlet drawer principle.

Corollary 1: A function f from a set with k+1 or more elements to a set with k elements is not one-to-one.

Proof: Suppose that for each element y in the codomain of f we have a box that contains all elements x of the domain f such that $f(x) = y$. Because the domain contains k+1 or more elements-and the co-domain contains only k elements, the pigeonhole principle tells us that one of these boxes contains two or more elements x of the domain. This means f cannot be one-to-one.

• Example 1 Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

The Generalized Pigeonhole Principle:

If N objects are placed into k boxes, then there is atleast one box containing at least $\lceil N/k \rceil$ objects.

Proof: We will use a proof by contradiction. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects.

Then the total number of object is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\left\lceil \frac{N}{k} \right\rceil + 1\right) - 1\right) = N$$

where the inequality $\lceil N/k \rceil < (N/k) + 1$ has been used.

This is a contradiction because there are total of N objects.

Example: Among 100 people there are atleast $\lceil \frac{100}{12} \rceil = 9$ who were born in the same month.

Example: How many cards must be selected from a standard deck of 52 cards to gurantee that atleast three cards of the same suit are chosen?

Solution: Pegion hole = 4   If atleast 3 pigeons should be in a hole
Pegion = ?

$\lceil \frac{N}{4} \rceil \geq 3$ $N = 2 \cdot 4 + 1 = 9$

Example: Find the minimum number of elements that needs to take from the set $A = \{1, 2, \cdots, 9\}$ to be sure that two of them add up to 10.

Solution:
Here, $A = \{1, 2, \cdots 9\}$ So pigeonholes are the five sets $(1, 9), (2, 8), (3, 7), (4, 6), (5, 5)$, So that we have $n = 5$ pigeonholes and $k + 1 = 2$ ~~...~~ (pigeons). Then $k = 1$. Now, $kn + 1 = 5 \times 1 + 1 = 6$.
Thus, any choice of six elements (pigeon) of A will gurantee that two of them add up to ten.

Example: How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these number add up to 16?

Solution: The pairs of numbers that sum 16 are $(1,15)$, $(3,13)$, $(5,11)$, $(7,9)$ i.e 4 pairs of numbers are there that add to 16. If we select 5 numbers then by Pigeonhole principle there are atleast $\lceil 5/4 \rceil = 2$ numbers, that are from the set of selected 5 numbers, that constitute a pair. Hence, 5 numbers must be selected.

Example: In a group of n people, there are two people who have an identical number of friends within the group. [Everyone has atleast one friend]

Solution:

n people.

no of friends $= \{1, \dots, n-1\}$

Pigeonhole $= n-1$

Pigeon $= n$

$$\left\lceil \frac{n}{n-1} \right\rceil = 2$$

Example:

PERMUTATIONS AND COMBINATIONS

PERMUTATION: A permutation of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of r elements of a set is called an r-permutation.

- Theorem:

If n is a positive integer and r is an integer with $1 \le r \le n$, then there are

$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$

- Corollary: If n and r are integers with $0 \le r \le n$, then $P(n,r) = \frac{n!}{(n-r)!}$

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Example: How many ways are there to select a first prize winner, a second prize winner and a third prize winner from 100 different people who have entered a contest?

Solution:

$n = 100$
$r = 3$

$$P(100,3) = \frac{n!}{(n-r)!} = \frac{100!}{97!} = 970,200$$

Combinations:

Combination of objects means just their collection without any regard to order or arrangement.

Theorem: The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals.

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Corollary: Let, n and r be non-negative integers with $r \le n$. Then.

$$C(n,r) = C(n,n-r)$$

Example: How many different committes of three students can be formed from a group of four students?

Soulution:

$n = 4$ $r = 3$ $C(n,r) = \frac{4!}{1 \times 3!} = 4$

Selection of three alphabets from a, b, c, d,

Combination	Permutations
abc	abc, acb, bac, bca, cab, cba
abd	abd, adb, bad, bda, dab, dba
acd	acd, adc, cad, cda, dac, dca
bcd	bcd, bdc, cbd, cdb, dbc, dcb

PERMUTATIONS WITH REPETITIONS

The permutations of n objects taken all at a time, when ~~theare~~ are p objects of one kind, q objects are of second kind, r objects or of a third kind, is

$$\frac{n!}{p!\,q!\,r!}$$

Example: How many seven ~~words~~-letter words can be formed using the letters of the word;

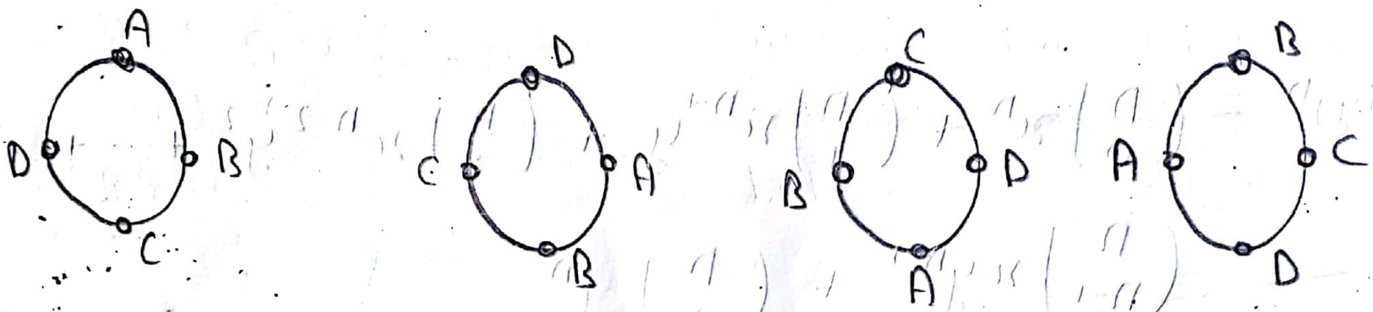
"BENZENE"?

Solution

$$\frac{7!}{3!\,2!} = 420$$

Circular Permutation:

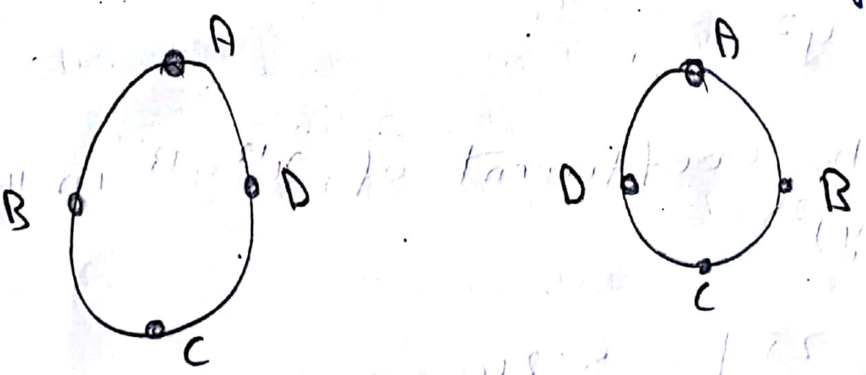
- The number of ways of arranging n unlike objects in a ring when clockwise and anticlock-wise arrangements are different is $(n-1)!$.



The above four arrangements are same as A always h D on his immediate right and B on his immediate left.

- The number of ways of arranging n unlike objects in a ring, when clockwise and anticlock wise arrangements are the same, is $\frac{(n-1)!}{2}$.

- Following two arrangements are the same - the one is the other viewed from the other side.



- The number of arrangements of 4 beads on a ring is $\frac{3!}{2}$.

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BINOMIAL THEOREM.

Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

or $(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \dots +$

$$\binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Example:

$(x+y)^2 = x^2 + 2xy + y^2$

Use Binomial Theorem

$$(x+y)^2 = \sum_{j=0}^{2} \binom{n}{j} x^{2-j} y^j$$

$$= \binom{2}{0} x^2 y^0 + \binom{2}{1} x^1 y^1 + \binom{2}{2} x^0 y^2$$

$$= x^2 + 2xy + y^2$$

Example What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

solution

$n = 25$

$j = 13$ $\binom{n}{j} = \binom{25}{13} = 5{,}200{,}300$

Example What is $(2a-b)^4$?

$$= \binom{4}{0}(2a)^4(-b)^0 + \binom{4}{1}(2a)^3(-b)^1 + \binom{4}{2}(2a)^2(-b)^2 + \binom{4}{3}(2a)^1(-b)^3 + \binom{4}{4}(2a)^0(-b)^4$$

$$1\times 16a^4 + 4\times 8a^3(-b) + 6\times 4a^2b^2 - 4\times 2ab^3 + 1\times b^4$$

$$= 16a^4 - 32a^3b + 24a^2b^2 - 8ab^3 + b^4$$

Pascal's Identity and Triangle:

Theorem
Let n and k be positive integers with $n \geq k$. Then.

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- Pascal's Identity, together with the initial conditions $\binom{n}{0} = \binom{n}{n} = 1$ for all integers n, can be used to recursively define binomial coefficients. This recursive definition is useful in the computation of binomial coefficients because only addition and not multiplication, of integers are needed to use this recursive definition.

Pascal's Identity is the basis for a geometric arrangement of the binomial coefficients in a triangle as shown in figure.

The n^{th} row in the triangle consists of the binomial coefficients.

$\binom{n}{k}$ $0, k = 0, 1, \ldots n$

This triangle is known as Pascal's Triangle.
Pascal's Identity shows that when two adjacent binomial coefficients in this triangle are added the binomial coefficients in the next row between these two coefficients is produced.

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\end{array}
$$

$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 1$

By Pascal's Identity

$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$

<u>Example</u>

$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

$\boxed{(x+y)^n \text{ takes coefficients of row } n+1}$

• <u>Generating Permutations and Combinations</u>

Suppose that a Salesman must visit six different cities. In which order should these cities be visited to minimize total travel time? One way to determine the best order is to determine the travel time for each of the $6! = 720$ different orders in which the cities can be visited and choose the one with the smallest ~~time~~ travel time.

• Second, suppose we are given a set of six positive integers and wish to find a subset of them that has 100 as their sum, if such a subset exist. One way to find these numbers is to generate all $2^6 = 64$ subsets and check the sum of their elements.

<u>Generating Permutations:</u>

Any set with n elements can be placed in one-to-one correspondence with the set $\{1, 2, 3, \ldots, n\}$. We can list the permutations of any set of n elements by generating the permutations of the n smallest positive integers and then replacing these integers with the corresponding elements.

• Here, we describe algorithm to generate permutation based on the lexicographic (or dictionary) ordering of the set of permutations of $\{1, 2, 3, \ldots, n\}$.

In this ordering, the permutation $a_1 a_2, \dots, a_n$ precedes the permutation of $b_1, b_2, \dots, b_n$ if for some k, with $1 \le k \le n$, $a_1 = b_1$, $a_2 = b_2$, $\dots$, $a_{k-1} = b_{k-1}$ and $a_k < b_k$.

Example: The permutation 23415 of the set {1, 2, 3, 4, 5} precedes the permutation 23514, because these permutation agree in the first two positions, but the number in the third position in the first permutation, 4 is smaller than the number in the third position in the second permutation, 5.

Similarly, the permutation

41532

precedes

52143

Example: What is the next permutation in lexicographic order after 362541?

Solution: The last pair of integers a_j and a_{j+1}, where $a_j < a_{j+1}$ is $a_3 = 2$ and $a_4 = 5$. The least integer to the right of 2 that is greater than 2 in the permutation is $a_5 = 4$. Hence, 4 is placed in the third position. Then the integers 2, 5 and 1 are placed in order in the last three positions, giving 125, as the last three positions of the permutation. Hence, the next permutation is 364125

Algorithm I: Generating the next permutation in Lexicographic Order

Procedure next permutation

$j = n-1$

while $a_j > a_{j+1}$

$j = j-1$

{ j is the largest subscript with $a_j < a_{j+1}$ }

$k = n$

while $a_j > a_k$

$k = k-1$

{ a_k is the smallest integer greater than a_j to the right of a_j }

interchange a_j and a_k

$r = n$

$s = j+1$

while $r > s$

begin

interchange a_r and a_s

$r = r-1$

$s = s+1$

end

{ This puts the tail end of the permutation after the jth position in increasing order }

<u>Generating Combinations</u>: How can we generate all the combinations of the elements of a finite set? Because a combination is just a subset, and has a 0 in this position if a_k is not in the subset. If all the bit strings of length n can be listed, then by the correspondence between subsets and bit strings, a list of all the subsets is obtained.

<u>Example</u>: Find the next bit string after
1000100111.

<u>Solution</u>:

1000101000

<u>Algorithm 2</u>: Generating next larger Bit String

procedure next_bit_string ($b_{n-1}b_{n-2}\cdots b_1b_0$: bit string not equal to $11\ldots11$)

```
i = 0
while, bi = 1
begin
    bi = 0
    i = i+1
end
bi = 1
```

Example: Find the next larger 4-combination of the set $\{1, 2, 3, 4, 5, 6\}$ after $\{1, 2, 5, 6\}$

Solution: The last term among the terms a_i with $a_1 = 1$, $a_2 = 2$, $a_3 = 5$ and $a_4 = 6$ such that $a_i = 6 - 4 + i$ is $a_2 = 2$. To obtain the next larger 4-combination, increment a_2 by 1 to obtain $a_2 = 3$. Then set $a_3 = 3+1 = 4$ and $a_4 = 3+2 = 5$. Hence, the next larger 4-combination is $\{1, 3, 4, 5\}$

1 1 0 0 1 1
1 1 0 1 0 0

Algorithm 3:

Generating the Next r-combination in Lexicographic Order.

Procedure: next r-combination ($\{a_1, a_2, \cdots, a_r\}$: proper subset of $\{1, 2, \ldots, n\}$ not equal to $\{n-r+1, \ldots, n\}$ with $a_1 < a_2 < \cdots a_r\}$.

$i = r$

while $a_i = n - r + i$

$i = i - 1$

$a_i = a_i + 1$

for $j = i+1$ to r

$a_j = a_i + j - i$

-ment is called a sample space. Each element of a sample space is called a sample point. The number of sample points in S may be denoted by n(S).

Eg. in the rolling the die, the sample space S={1,2,3,4,5,6} and sample points may be 1 or 3.

- **Event:** The results or outcomes of experiments are called events.

- **Equally like events:**
A number of events are said to be equally likely if any one of them cannot be expected to occur in prefence to the other.

- **Exhaustive events:** Events are said to be exhaustive when they include all possible outcomes of a random experiment. In tossing a coin, exhaustive events are two.

Mutually Exclusive Events: Two or more events are said to be mutually exclusive or disjoint if the events cannot occur simultaneously i.e. occurence of one of the events prevents the occurence of others.

Let, A and B are any two events defined on a sample space S and $A \cap B = \phi$ then A and B are called mutually exclusive.

Eg. While tossing a coin the events occurence of head and tails are mutually exclusive.

Probability of Events:

$$P(E) = \frac{\text{Number of outcomes favourable to the occurence of } E}{\text{Total number of all possible outcomes.}}$$

$$P(E) = \frac{n(E)}{n(S)}$$

The probability of non-occurence of the event A is

$$P(E') = \frac{n(S) - n(E)}{n(S)} , \quad P(E') = 1 - P(E)$$

Example: Three unbiased coin are tossed

ⓐ Write the sample space.

ⓑ Find the probability of: all heads, at most 2 heads.

Solution:

(a) Let S denotes the sample space of tossing three coins.

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

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Additional Principle:

For any events A and B.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example:

Suppose a student is selected at random from 200 students, when 70 are taking C-programming, 50 are taking JAVA and 20 are taking both. Find the probability that the student is taking C or Java.

Solution.

$$P = P(C \cup J) = P(C) + P(J) - P(C \cap J)$$

Conditional Probability:

Let E be an event in a sample space S with $P(E) > 0$ then probability then an event A occurs once E has occured or specifically, the conditional probability of A given E, written as $P(A/E)$, is defined as.

$$P(A/E) = \frac{P(A \cap E)}{P(E)}$$

Example:

What is the conditional probability that a randomly generated bit string of length four contains at least two successive consecutive 0s, given the first bit is 1?

(Assume that probability of '0' and '1' are same).

Solution:
Let E be the event that a bit string of length four contains at least 0s and F be the event that the first bit is 1. The probability that a bit string of length four has at least two consecutive 0s, given that its first bit is 1 equals $P(E/F)$.

Without restriction the number of ways of bit string of length four can be formed with digits 0 and 1 is $2^4 = 16$.

There are $2^3 = 8$ ~~ways~~ bit strings of length four that start with 1.

Thus, $P(F) = \frac{8}{16} = \frac{1}{2}$

Again, $E \cap F = \{1000, 1100, 1001\}$

So $P(E \cap F) = \frac{3}{16}$

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{3/16}{8/16} = 3/8$$

Independence: Events A and B in a probability space S are said to be independent if the occurence of one of them does not influence the occurence of the other.

In other words, B is independent of A is $P(B)$ is same as ~~$P(A)$~~ $P(B/A)$.

i.e events A and B are independent if $P(A \cap B) = P(A) \cdot P(B)$

Ex: In tossing two coins, the appearence of head one coin does not affect the appearence of head in other coin.

Ex Suppose A is the event that a randomly generated bit string of length four begins with 1 and B is the event that this bit contains an even number of ones. Are A and B independent, if the 16 bit strings of length four are equally likely?

Solution:

There are eight bit strings of length four that begin with one: 1000, 1001 - - -

There are also eight bit strings of length four that contain even number of ones.

$$P(A) = \frac{8}{16} = \frac{1}{2}$$

$$P(B) = \frac{8}{16} = \frac{1}{2}$$

$$P(A \cap B) = \{1111, 1100, 1010, 1001\}$$

$$\therefore P(A \cap B) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(A) \cdot P(B)$$

Hence, A and B are independent.

Random variable:

A variable whose numerical value is determined by the outcome of a random experiment is called a random variable.

A random variable X is a real valued function, $X(x)$, of the elements of the sample space S where x is an element of the sample space. Random variables assigns a real number to each possible outcome.

Example: If we toss a coin and denote the head by 1 and tail by 0, then the random variable X takes only two values 1 and 0. Symbolically, the random variable,

$$X(x) = \{x : x = (1,0) \in S\}$$

Expected Value and Variance:

The expected value of random variable is the sum of overall elements in a sample space of the product of the probability of the element and the value ~~of the val~~ of the random variable at this element.

The expected value of a random variable provides a central point for the distribution of values of this random variables.

Another useful measure of a random variable is its variance, which tells us how spread out the values of this random variable are.

$\therefore$ The expected value of the random variable $X(s)$ on the sample space S is equal to

$$E(X) = \sum_{s \in S} p(s)\, X(s)$$

Example: Let X be the number that comes up when a die is rolled. What is the expected value of X?

Solution:

The random variable X takes the values $1, 2, 3, 4, 5$ and 6, each of them has probability $1/6$.

This results: $E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6$

$$= \frac{7}{2}$$

Example: An unbiased coin is tossed three times. Let S be the sample space of eight possible outcomes and X be the random variable that assigns to an outcome the number of tails in this outcome.

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What is the expected value of X?

Solution:

When an unbiased coin is tossed three times, the eight possible outcomes in sample space are.

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Since coin is unbiased and toss are independent then the probability of each outcome is $\frac{1}{8}$.

$$\therefore E(X) = \frac{1}{8}[X(HHH) + X(HHT) + X(HTH) + X(HTT) + X(THH) + X(THT) + X(TTH) + X(TTT)]$$

$$= \frac{1}{8}[0+1+1+2+1+2+2+3]$$

$$= \frac{3}{2}$$

Variance: Variance of a random variable helps us characterize how widely a random variable is distributed.

Let X be a random variable on a sample space S. The variance of X, denoted by V(X) is

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s)$$

$$V(X) = E(X^2) - E(X)^2$$

$V(X)$ is the weighted average of the square of the deviation of X.
The standard deviation of X, denoted by $\sigma(X)$ is defined to be $\sqrt{V(X)}$.

Example: What is the variance of the random variable X, where X is the number that comes up when a die is rolled?

Solution:

We have, $V(X) = E(X^2) - E(X)^2$

The random variable X takes the values 1, 2, 3, 4, 5 or 6, each of them has probability $1/6$.
This results: $E(X) = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 = 7/2$

Since. X^2 takes the values i^2, $i = 1, 2, \ldots 6$, each with probability $1/6$.
It follows that

$$E(X^2) = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \frac{91}{6}$$

Hence, $V(X) = \frac{91}{6} - (7/2)^2$

$$= \frac{35}{12}$$

$$\sigma(X) = \sqrt{V(X)} = \sqrt{\frac{35}{12}} =$$

Randomized Algorithm:

An algorithm that uses random numbers to decide what to do next anywhere in its logic is called Randomized Algorithm.

Two types:

I: Las Vegas algorithm: It uses random input so that they always terminate with the correct answer, ~~but where the expected running time is finite.~~

II. Monte Carlo Algorithm: These algorithms have chance of producing an incorrect result or fail to produce a result either by signaling a failure or failing to terminate.

Types

Las Vegas Algorithm: it is a randomized algorithm that always gives correct result or it informs about the failure.

A las vegas algorithm does not gamble with correctness of the result; it gambles only with the resources used for the computation.

Eg. Randomized Quick sort, where pivot is choosen randomly, but result is always sorted.

Monte Carlo Algorithm: Answers may be incorrect with certain (typically small) probability and the resources used are bounded.
eg. Karger-Stein algorithm.

Primality Test: A primality test is an algorithm for determining whether an input number is prime.

Probability Primality Test:

Miller-Robin PRrimality Test:

General Algorithm

1) Enter n
2) find $n-1 = 2^k \cdot m$
3) Choose, a; $1 \leq a < n-1$
4) Compute

$b_0 = a^m \pmod{n}$ if $+1$ / -1 } Probably Prime
else goto step 5

5) for $i=1$ to $i=k-1$

$b_i = b_{i-1}^2 \pmod{n}$

if $+1$ implies composite, end loop
if -1 probably prime, end loop

Example:

(i) $n = 53$

(ii) $n-1 = 52$, $52 = 2^2 \times 13$, $k=2$, $m=13$

(iii) Let, $a = 2$

(iv) $b_0 = 2^{13} \pmod{53} = 30$

Since, it is not $+1$ or -1

we go to step 5.

(v) $b_1 = b_0^2 \pmod{53} = 52 = -1 \bmod 53$

$\therefore b_1 = -1 \bmod 53$

↳ n is probably prime

- Now, we take different value of 'a' and do this process again.

- The probability that a composite integer n passes Miller's test for a randomly selected base a, is less than $1/4$.

Because the integer a with $1 < a < n$ is selected at random at each iteration and these iterations are independent, the probability that n is composite but the algorithm responds that n is prime is less than $(1/4)^m$.

By taking m to be sufficiently large, we can make this probability extremely small.

ADVANCED COUNTING

Recurrence Relations:

Suppose $a_0, a_1, a_2, \dots a_n$ is a sequence. A recurrence relation for the n^{th} term a_n is a formula (i.e function) giving a_n in terms of some or all previous terms (i.e $a_0, a_1, \dots a_n$). To find the ~~the~~ complete sequence, the rest few initial values are needed. These initial values are called initial conditions.

Ex $a_n = a_{n-1} - a_{n-2}$, $a_0 = 3$ & $a_1 = 5$

Sol.

$a_2 = 5 - 3 = 2$

$a_3 = 2 - 5 = -3$

Solving Recurrence Relation:

If the given recurrence relation involving sequence $a_0, a_1, a_2, \dots a_n$ then the solution of such recurrence relation is to find an explicit formula for general term a_n.

Example:

Solve the recurrence relation

$$a_n = 2a_{n-1}, \ n \geq 1 \text{ and } a_0 = 3$$

Solution:

$a_0 = 3$

$a_1 = 2 \times 3 = 2(3)$

$a_2 = 2a_1 = 2(2 \times 3) = 2^2(3)$

$a_3 = 2a_2 = 2(2^2 \times 3) = 2^3(3)$

$\vdots$

$a_n = 2^n(3)$

Hence, the general solution for given recurrence relation is, $a_n = 2^n(3)$

Example:

Solve the recurrence relation $a_n = a_{n-1} + 2$ Subject to initial condition, $a_1 = 3$.

Solution:

$a_n = a_{n-1} + 2$

$\boxed{a_2 = 3 + 2}$

$a_n = (a_{n-2} + 2) + 2 = a_{n-2} + 4$

$a_n = (a_{n-3} + 2) + 4 = a_{n-3} + 6$

$a_n = (a_{n-4} + 2) + 6 = a_{n-4} + 8$

$\vdots$

$a_n = a_{n-(n-1)} + 2 \times (n-1)$, $a_n = a_1 + 2(n-1)$

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$a_n = 3 + \cancel{3}2(n-1)$

Solving Linear Recurrence Relations.

- A linear homogeneous recurrence relation of degree k with ~~s~~ constant coefficients is recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

where $c_1, c_2, \dots, c_k$ are real numbers and $c_k \neq 0$.

- This recurrence relation in the definition is linear because the right-hand side is a sum of previous terms of the sequence each multi-plied by a function of n.
- The recurrence relation is homogeneous because no terms occur that are not multiple of the a_js.
- The coefficients of the terms of the sequence are all constants, rather than functions that depend on n.
- The degree is k because a_n is expressed in terms of the previous k terms of the sequence.

Ex

$P_n = (1.11) P_{n-1}$ → a linear homogeneous of degree 1

$f_n = f_{n-1} + f_{n-2}$ → linear homogeneous of degree 2

$a_n = a_{n-5}$ → " " " " 5

$a_n = a_{n-1} + a_{n-2}^2$ → not linear

$H_n = 2H_{n-1} + 1$ → not homogeneous

$B_n = nB_{n-1}$ → does not have constant coefficients.

Theorem I:

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$ where α_1 and α_2 are constants.

Example 3: What is the solution of the recurrence relation

$a_n = a_{n-1} + 2a_{n-2}$

with $a_0 = 2$ and $a_1 = 7$?

Solution:

We've $a_n = a_{n-1} + 2a_{n-2}$

Comparing with,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

$c_1 = 1 \quad c_2 = 2$

Now, we have characteristic equation of the recurrence relation in form.

$$r^2 - c_1 r - c_2 = 0$$

Substituting c_1 and c_2; we get

$$r^2 - r - 2 = 0. \quad -(i)$$

Solving (i)

$r = 2$, $r = -1$

Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n$$

for some constants α_1 and α_2.

From the initial conditions it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2$$

$$a_1 = 7 = 2\alpha_1 - \alpha_2$$

Solving, we get, $\alpha_1 = 3$ & $\alpha_2 = -1$

Hence, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$$a_n = 3 \cdot 2^n - (-1)^n$$

Example: Find an explicit formula for the Fibonacci numbers

Sol.

$$f_n = f_{n-1} + f_{n-2} \quad \text{— (i)}$$
$$f_0 = 0$$
$$f_1 = 1$$

Comparing equation (i) with

$$f_n = c_1 f_{n-1} + c_2 f_{n-2} + \dots + c_k f_{n-k}$$

$c_1 = 1$, $c_2 = 1$

$\therefore \; r^2 - c_1 r - c_2 = 0$

or $r^2 - r - 1 = 0$

Solving we get $\quad r_1 = \left(\frac{1+\sqrt{5}}{2}\right) \quad r_2 = \left(\frac{1-\sqrt{5}}{2}\right)$

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1+\sqrt{5}}{2}\right)^n$$

for some constants α_1 and α_2. The initial conditions $f_0 = 0$ and $f_1 = 1$ can be used to find these constants.

$$f_0 = \alpha_1 + \alpha_2 = 0$$
$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n = 1$$

Solving we get

$$\alpha_1 = \frac{1}{\sqrt{5}} \qquad \alpha_2 = -\frac{1}{\sqrt{5}}$$

Consequently, the Fibonacci numbers are given by

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

- Theorem I does not apply when there is one characteristic root of multiplicity two.

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2 \dots$ where α_1 and α_2 are constants.

Example: What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions $a_0 = 1$ & $a_1 = 6$?

Solution:

$c_1 = 6 \quad c_2 = -9$

$r^2 - c_1 r - c_2 = 0$

$\therefore r^2 - 6r + 9 = 0$

$\underline{r = 3}$

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Hence, solution to this recurrence relation is

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

for some constants α_1 and α_2

$a_0 = 1$ $a_1 = 6$.

$\alpha_1 = 1$

$a_0 = 1 = \alpha_1$

$\alpha_2 = 1$

$a_1 = 6 = 3 + \alpha_2 3$

$\alpha_2 = 1$

$\therefore$ $a_n = 3^n + n3^n$

- The following theorem (Theorem 3) states the general result about the solution of linear homogeneous recurrence relations with constant coefficients, where the degree may be greater than two, under the assumption that the characteristic equation has distinct roots.

- **Theorem 3:**

Let $c_1, c_2, \ldots, c_k$ be a real numbers. Suppose that the characteristic equation.

$$r^k - c_1 r^{k-1} \cdots - c_k = 0$$

has k distinct roots, $r_1, r_2, \ldots, r_k$. Then the sequence $\{a_n\}$ is the solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$ where $\alpha_1, \alpha_2, \ldots \alpha_k$ are constants.

Example: Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions

$$a_0 = 2 \quad a_1 = 5, \quad a_2 = 15$$

Solution

$$c_1 = 6 \quad c_2 = -11 \quad c_3 = 6$$

The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6 = 0$$

Solving, ~~we~~ $(r-1)(r-2)(r-3) = 0$

$$\therefore r = 1, 2, 3$$

Hence the solutions to the recurrence relations are of the form

$$a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n$$

To find the constants we use the initial value

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$$

$$a_1 = 5 = \alpha_1 + 2\alpha_2 + 3\alpha_3$$

$$a_2 = 5 = \alpha_1 + 4\alpha_2 + 9\alpha_3$$

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Solving we get

$\alpha_1 = 1$

$\alpha_2 = -1$

$\alpha_3 = 2.$

$\therefore\ a_n = 1 - 2^n + 2 \cdot 3^n$

- The following theorem is for linear homogeneous recurrence relations with coefficients, allowing the characteristic equation to have multiple roots.

- <u>Theorem 4:</u>

Let $c_1, c_2 \ldots c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$.

Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$$

for $n = 0, 1, 2 \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Example 8: Find the solution to the recurrence solution.

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$, initial condition $a_0 = 1, a_1 = -2, a_2 = -1$

Solution:

$C_1 = -3 \quad C_2 = -3 \quad C_3 = -1 \quad , \quad k = 3$

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_{k-1} r - c_k = 0$$

$$r^3 + 3r^2 + 3r + 1 = 0$$

$$(r+1)^3 = 0$$

$$r = -1$$

There is a single root $r = -1$ of multiplicity three of the characteristic equation.

From theorem 4,

$$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1} n(-1)^n + \alpha_{1,2} n^2 (-1)^n$$

To find the constants $\alpha_{1,0}, \alpha_{1,1}$ and $\alpha_{1,2}$ use the initial conditions.

$$a_0 = 1 = \alpha_{1,0}$$

$$a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2}$$

$$a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}$$

Solving we get,

$\alpha_{1,0} = 1, \quad \alpha_{1,1} = 3 \quad$ and $\alpha_{1,2} = -2$

$$\therefore \quad a_n = (1 + 3n - 2n^2)(-1)^n$$

Linear ~~homo~~ nonhomogeneous Recurrence with Constant Coefficients.

eg. $a_n = 3a_{n-1} + 2n$.

<u>Theorem 5</u>: If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

<u>Example</u> Find all solutions of the recurrence relation, $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

<u>Solution</u>:

To solve this linear nonhomogeneous relation with constant coefficients, we need to solve its associated linear homogeneous equation and to find a particular solution for the given non homogeneous equation.

For homogeneous equation we have.

$a_n = 3a_{n-1}$ or $a_n - 3a_{n-1} = 0$

$r - 3 = 0$ or $r = 3$

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$a_n^{(h)} = \alpha r^n$

or $a_n^{(h)} = \alpha 3^n$, where α is a constant.

We now find a particular solution.

Because $F(n) = 2n$ is a polynomial in n of degree one, a reasonable trial solution is a linear function in n,

Say,

~~$P_n = Cn + d$~~ , where c and d are constants.

$P_n = Cn + d$

~~$a_n = 3a_{n-1} + 2n$~~ $a_n = 3a_{n-1} + 2n$

~~$Cn + d + 3(C(n-1) + d)$~~ $Cn + d = 3(c(n-1) + d) + 2n$

Simplifying and combining like terms we get

$(2 + 2c)n + (2d - 3c) = 0$

It follows that $Cn + d$ is a solution if and only if $2 + 2C = 0$ and $2d - 3c = 0$

This shows that $Cn + d$ is solution if and only if,

$c = -1$

$d = -3/2$.

Consequently,

$a_n^{(p)} = -n - 3/2$

where, α is a constant

$a_n = a_n^{(p)} + a_n^{(h)}$

$a_n = -n - \frac{3}{2} + \alpha 3^n$,

We have,

$a_1 = 3$

for $n=1$, $a_n = 3$

$3 = -1 - \frac{3}{2} + \alpha \times 3$

$4 + \frac{3}{2} = 3\alpha$

$\frac{11}{2} = 3\alpha$, $\alpha = 11/6$

$\therefore$ $a_n = -n - \frac{3}{2} + (11/6) 3^n$.

<u>Example:</u> Find all solutions of the recurrence relation.

$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

<u>Solution:</u>

$a_n = a_n^{(h)} + a_n^{(p)}$

Solving for $a_n^{(h)}$

$a_n = 5a_{n-1} - 6a_{n-2}$

$r^2 - 5r + 6 = 0$

$r^2 - 3r - 2r + 2 \cdot 3 = 0$ or $r(r-3) - 2(r-3)$

$(r-3)(r-2) = 0$ or $r = 3, 2$

$\therefore$ $a_n^{(h)} = \alpha_1 (r_1)^n + \alpha_2 (r_2)^n$

$a_n^{(h)} = \alpha_1 3^n + \alpha_2 2^n$

- Because $F(n) = 7^n$, a reasonable trial solution is $a_n^{(p)} = C \cdot 7^n$, where C is a constant.

Substituting the terms of this sequence into the recurrence relation implies that.

$$C \cdot 7^n = 5 \cdot C 7^{n-1} - 6 C 7^{n-2} + 7^n$$

$$C 7^2 = 5 C 7 - 6 C + 7^2$$

or $49C = 35C - 6C + 49$

or $C = 49/20$

Hence, $a_n^{(p)} = (49/20) 7^n$.

$\therefore$ $a_n = a_n^{(h)} + a_n^{(p)}$

or $a_n = \alpha_1 \cdot 3^n + \alpha_2 2^n + \left(\frac{49}{20}\right) 7^n$

Example: Find all solutions of the recurrence relation $a_n = 4 a_{n-1} + n^2$. Also find the relation with initial condition $a_1 = 1$.

Solution

$$a_n = 4a_{n-1} + n^2$$

$$a_n = a_n^{(h)} + a_n^{(p)}$$

for $a_n^{(h)}$

$$a_n - 4a_{n-1} = 0$$

or $r - 4 = 0$, $r = 4$

$\therefore$ $a_n^{(h)} = \alpha (r)^n$

or $a_n^{(h)} = \alpha (4)^n$, where α is constant.

- Since $F(n) = n^2$ is a polynomial of degree 2, a trial solution is a quadratic function in n, say;

$$P_n = an^2 + bn + c$$, where a, b and c are constants

To determine whether there are any solution of this form, suppose that

$P_n = an^2 + bn + c$, is such solution

Then the equation $a_n = 4a_{n-1} + n^2$, becomes

$$an^2 + bn + c = 4\left(a(n-1)^2 + b(n-1) + c\right) + n^2$$

$$an^2 + bn + c = 4\left(a(n^2 - 2n + 1) + bn - b + c\right) + n^2$$

$$an^2 + bn + c = 4(an^2 - 2an + a + bn - b + c) + n^2$$

$$an^2 + bn + c = (4an^2 + n^2) - 8an + 4bn + 4(a - b + c)$$

$$(a - 4a - 1)n^2 + (b + 8a - 4b)n + \underset{c-4c}{\cancel{c}} - 4a + 4b = 0$$

$$(-3a - 1)n^2 + (8a - 3b)n + (-4a + 4b) = 0$$

$-3c$

(ii)

- Here an^2+bn+c ~~to~~ be the solution

$\therefore$ $~~6~~-3a-1=0$ $\quad\quad$ $8a-3b=0$

$a=-1/3$ $\quad\quad$ $b=-8/9$

$-4a+4b-3c=0$

$-4\times\frac{-1}{3}+4\times\frac{-8}{9}=3c\ ,\quad \frac{4}{3}-\frac{32}{9}=3c$

$-\frac{20}{9}=3c\ ,\ c=-\frac{20}{27}$

So $a_n^{(p)}=\frac{-1}{3}n^2-\frac{8}{9}n-\frac{20}{27}$

is a particular solution.

$a_n=a_n^{(h)}+a_n^{(p)}$

$a_n=\frac{-1}{3}(n^2+\frac{8}{3}n+\frac{20}{9})+\alpha 4^n$

For solution with $a_1=1$

$n=1,\ a_n=1$

$1=-\frac{1}{3}(1+8/3+\frac{20}{9})+\alpha 4$

$1+\frac{1}{3}=\frac{9+24+20}{9}+\alpha 4$

$\frac{4}{3}-\frac{53}{9}=\alpha 4$

$\frac{12-53}{9\times 4}=\alpha$ $\quad$ or $\alpha=-41/36$

$\therefore a_n=-\frac{1}{3}(n^2+\frac{8}{3}n+\frac{20}{9})+(-41/36)4^n$

Example. Find all solution of recurrence relation $a_n = 2a_{n-1} + 3^n$ and solution with initial condition $a_1 = 5$.

Solution:

$$a_n = 2a_{n-1} + 3^n$$

$$a_n = a_n^{(h)} + a_n^{(p)}$$

Solving for $a_n^{(h)}$

$a_n - 2a_{n-1} = 0$ or $r - 2 = 0$ $r = 2$.

$a_n^{(h)} = \alpha r^n$ or $a_n^{(h)} = \alpha 2^n$

For the particular solution we write

$$F(n) = 3^n$$

Let, the trial solution be

$P(n) = c \cdot 3^n$, where c is a constant

Substituting terms of this sequence into recurrence relation, we get

$$c\,3^n = 2c\,3^{n-1} + 3^n$$

$$3c = 2c + 3$$

$$c = 3$$

$\therefore$ Particular solution is

$$a_n^{(p)} = 3 \cdot 3^n = 3^{n+1}$$

$$a_n = a_n^{(p)} + a_n^{(h)} = \alpha 2^n + 3^{n+1}$$

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$a_1 = 5, \quad n = 1, \quad a_n = 5$

$5 = \alpha\, 2^1 + 3^2$

$5 = 2\alpha + 9$ or $\frac{-4}{2} = \alpha$, or $\alpha = -2$

Hence the solution is

$a_n = (-2)\, 2^n + 3^{n+1}$

$a_n = -2^{n+1} + 3^{n+1}$

DIVIDE AND CONQUER ALGORITHM

- A Divide and conquer is an algorithm design paradigm based on multi-branched recursion. A divide and conquer algorithm works by recursively breaking down a problem into two or more sub-problems of same or related type, until these becomes simple enough to be solved directly.

Divide and Conquer Recurrence Relations:

Suppose that a recursive algorithm divides a problem of size 'n' into 'a' subproblems, where each subproblem is of size n/b. Also, suppose that a total of g(n) extra operations are required in the conquer step of the algorithm to combine the solutions of a subproblems into a solution of the original problem.

Then, if $f(n)$ represents the number of operations required to solve the problem of size n, it follows that f satisfies the recurrence relation.

$$f(n) = af(n/b) + g(n)$$

This is called divide and conquer recurrence relation.

- **Example: Binary Search:**

The binary search algorithm reduces the search for an element in a search sequence of size n to the binary search for this element in a search sequence of n/2, when n is even. Two comparisons are needed to implement this reduction.

Hence, if $f(n)$ is the number of comparisons required to search for an element in a a search sequence of size n, then.

$$f(n) = f(n/2) + 2$$

when, n is even.

- Merge Sort:

The merge sort algorithm splits a list to be sorted with n items, where n is even, into two lists with $n/2$ elements each, and uses fewer than n comparisons to merge the two sorted lists of $n/2$ items each into one sorted list.

Consequently, the number of comparisons used by the merge sort to sort a list of n elements is ~~a~~ less than $M(n)$, where $M(n)$ satisfies the divide-and-conquer recurrence relation.

$$M(n) = 2M(n/2) + n.$$