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UNIT 3: Logic and Proof Methods

- Propositional Logic:

- A Proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false but not both.

Example

1. Ankara is the capital of Turkey.
2. Sydney is the capital of Australia.
3. $1+1=3$
4. $2+5=7$

Propositions 1 and 4 are true, while 3 and 2 are false.

Example:

1. What time is it?
2. Read this carefully.
3. $x+1=2$
4. $x+y=z$

- The area of logic that deals with propositions is called the propositional calculus or propositional logic.

Definition 1: Let p be a proposition. The negation of p, denoted by $\neg p$ (also denoted by $\bar{p}$), is the statement

"It is not the case that p".

The proposition $\neg p$ is read "not p". The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

Example: Find the negation of the proposition
"Today is Friday"
and express this in simple English.

Solution: The negation is
"It is not the case that today is Friday".

OR simply
"~~Today~~ Today is not Friday".

Example: Find the negation of the proposition.
"At least 10 inches of rain fell today in Miami".

Solution: The negation is
"It is not the case that at least 10 inches of rain fell today in Miami".

OR.

"Less than 10 inches of rain fell today in Miami."

Truth Table: The truth table for the negation of a proposition.

p	$\neg p$
T	F
F	T

Definition 2: Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "p and q". The conjuction $p \wedge q$ is true when both p and q are true and is false otherwise.

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Truth table: The truth table for the conjuction of two propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: Find the conjuction of the propositions p and q where p is the proposition, "Today is Friday" and q is the proposition, "It is raining today".

Solution: The conjuction of these propositions, $p \wedge q$, is the proposition "Today is Friday and it is raining today."
This proposition is true on rainy Fridays and is false on any day that is not ~~s~~ Friday and on Fridays when it does not rain.

Definition 3: Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q". The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Example: Find the disjunction of the propositions p and q where p is the proposition "Today is Friday" and q is the proposition "It is raining".

Solution: The disjunction of p and q, p ∨ q is the proposition.
"Today is Friday or it is raining today."

This is true on any day that is either a Friday or a rainy day (including rainy Fridays). It is only false on days that are not Fridays when it is also does not rain.

Definition 4: Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

~~Definition:~~
Table: The truth table for the 'exclusive or' of two proposition.

p	q	$p \oplus q$

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Definition 5: Let p and q be propositions. The conditional statements $p \to q$ is the proposition "if p, then q". Then conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

- The statement $p \to q$ is called a conditional statement because $p \to q$ asserts that q is true on condition that p holds. The conditional statement is also called implication.

- **Truth table:** The truth table for the conditional statement $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Example:** "If I am elected, then I will lower taxes"

p "Maria learns discrete mathematics".

q "Maria will find a good job"

$p \to q$ "Maria learns discrete mathematics then she will find a good job."

- This statement is true unless Maria learns discrete mathematics, but she does not get a good job.

- "If today is Friday, then $2+3=5$" is true from the definition of a conditional statement, because its conclusion is true.

- "If today is Friday, then $2+3=6$" is true every day except Friday, even though $2+3=6$ is false.

- Converse, Contrapositive and Inverse.

- The proposition $q \to p$ is called the converse of $p \to q$

- The contrapositive of $p \to q$ is the proposition $\neg q \to \neg p$.

- The proposition $\neg p \to \neg q$ is called inverse of $p \to q$.

- Only $\neg q \to \neg p$ (contrapositive) has the same truth value as $p \to q$.
- The contrapositive is false only when $\neg p$ is false and $\neg q$ is true, that is, only when p is true and q is false.

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ruth table.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$q \to p$
T	T	T
T	F	T
F	T	F
F	F	T

p	q	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

p	q	$\neg p$	$\neg q$	$\neg p \to \neg q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Solution:

$p \to q$ "If it is raining, then the home team wins."

- Consequently, the contrapositive of this conditional statement is.

"If the home team does not wins, then it is not raining."

- ~~The~~ The converse is.

"If the home team wins, then it is raining"

- Then inverse is

"If it is not raining, the home team doesn't win."

Biconditionals

Definition: Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q". The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications. $p \leftrightarrow q$ has exactly the same truth values as $(p \to q) \wedge (q \to p)$.

Example: Let p be the statement "You can take the flight" and let q be the statement "You buy a ticket". Then $p \leftrightarrow q$ is the statement "You can take a flight if and only if you buy a ticket".

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of logical operators:

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Translating English Sentences:

Example How can this English sentence be translated into a logical expression?

"You can access the internet from campus only if you are a computer science major or you are not a freshman".

Solution:

'a' = You can access Internet from campus

'b' = You are a computer science major

'c' = You are a freshman.

$\neg c$ = You are not a freshman

$b \vee \neg c$ = You are a computer science major or you are not a freshman.

$a \rightarrow (b \vee \neg c)$

Example: Express the specification "The automated reply cannot be sent when the file system is full" using logical connectives.

Solution:

a = The automated reply can be sent

b = The file system is full.

$\neg a$ = The automated reply cannot be sent

$b \rightarrow \neg a$.

Propositional Equivalences.

Definition: A compound proposition that is always true, no matter what the truth values of the propositions that occur in it, is called a tautology. A compound proposition that is always false is called a contradiction. A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Examples

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Fig: Examples of a tautology and contradiction

Definition 2. The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.

Example Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution:

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$

Example: Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

- **Using De Morgan's Law**
  - $\neg(p \vee q) \equiv \neg p \vee \neg q$
  - $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Example: Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer."

→ p = "Miguel has a cellphone"
q = "He has a laptop"

$p \wedge q$ = "Miguel has a cellphone and he has a laptop."

$\neg(p \wedge q)$ = It is not the case that, Miguel has a cell phone and he has a laptop.

$\neg p$ = Miguel does not have a cell phone.
$\neg q$ = He does not have a laptop.

~~$p \vee q$~~ $\neg p \vee \neg q$ = Miguel does not have a cell phone or he does not have a laptop.

Example: Show that $\neg(p \to q)$ and $p \to \neg q$ are logically equivalent.

Example: Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalence.

Solution:

$\neg(p \vee (\neg p \wedge q)) = \cancel{\neg p \wedge \neg(p \wedge q)}$

$= \neg p \wedge \neg(\neg p \wedge q)$. by the second De Morgan law.

$= \neg p \wedge (p \vee \neg q)$. by the first De Morgan law.

$= (\neg p \wedge p) \vee (\neg p \wedge \neg q)$. by second distributive law.

$= F \vee (\neg p \wedge \neg q)$. because $\neg p \wedge p = F$

$= (\neg p \wedge \neg q) \vee F$. by commutative law for distribution.

$= \neg p \wedge \neg q$. by identity law for F

Example Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

See book. page # 24. for tables.

Predicates and quantifiers.

Predicates: Statements involving variables such as.
"$x>3$", "$x=y+3$", "$y+x=z$"
" Computer x is under attack by an intruder".
and
" computer x is functioning properly".

- The above statements are neither true nor false when the values of the variables are not specified.

- The statement "x is greater than 3" has two parts. The first part, the variable x, is the subject of the statement. The second part - the predicate, "is greater than 3" - refers to the property that the subject of the statement can have.
- We denote the statement "x is greater than 3" by $P(x)$, where P denotes the predicate "is greater than" and x is a variable.
- The statement $P(x)$ is said to ~~have~~ be the value of the propositional function P at x. Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

Example: Let $P(x)$ denote the statement "$x>3$". What are the truth values of $P(4)$ and $P(2)$?

Solution:

$P(4) =$ True

$P(2) =$ False.

Example: Let $Q(x,y)$ denote the statement "$x = y+3$". What are the truth values of the propositions $Q(1,2)$ and $Q(3,0)$?

Solution:

$x=1$, $y=2$, $Q(1,2) = F$

$1 = 2+3$

$\downarrow$

F

$x = 3$

$y = 0$

$3 = 0+3$

$\downarrow$

T

$Q(3,0) = T$

- A statement involving the n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$. A statement of the form $P(x_1, x_2, \ldots x_n)$ is the value of the propositional function P ~~at~~ the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is also called an n-place predicate or a n-ary ~~[illegible]~~ predicate.

Example: Consider the statement.

if $x>0$ then $x := x+1$

· Here, $P(x)$ is "$x>0$"

when $P(x)$ is true $x := x+1$ is executed.

- The statements that describe valid input are known as preconditions.
- The conditions that the output should satisfy when the program has run are known as postconditions.

the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall x$ is called universal quantifier. We read $\forall x P(x)$ as "for all $x P(x)$" or "for every $x P(x)$". An element for which $P(x)$ is false is called a counterexample of $\forall x P(x)$.

<u>Definition</u>: The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$".

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

Statement	When True?	When False?
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false
$\exists_x P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x.

Example: Let $P(x)$ be the statement "$x+1 > x$". What is truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Solution: Because $P(x)$ is true for all real numbers x, the quantification $\forall x P(x)$ is true.

Example: What is the truth value of $\forall x (x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

Solution: The universal quantification $\forall x (x^2 \geq x)$, where the domain consists of all real numbers, is false.

For example, $(1/2)^2 \geq 1/2$.

Note that $x^2 \geq x$ if and only if $x^2 - x = x(x-1) \geq 0$. Consequently, $x^2 \geq x$ if and only if $x \leq 0$ or $x \geq 1$. It follows that $\forall x (x^2 \geq x)$ is false if the domain consists of all real numbers (because the inequality is false for all real numbers x with $0 < x < 1$).

- However, if the domain consists of the integers $\forall x (x^2 \geq x)$ is true, because there are no integers x with $0 < x < 1$.

Example Let $Q(x)$ denote the statement "$x = x+1$". What is truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers.

Solution: Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists x\, Q(x)$, is false.

Example: What is truth value of $\exists x\, P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution: Because the domain is $\{1, 2, 3, 4\}$. the proposition $\exists x\, P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4)$$

Because $P(4)$, which is the statement "$4^2 > 10$" is true, it follows that $\exists x\, P(x)$ is true.

- When all the elements in the domain can be listed - say, $x_1, x_2, \ldots, x_2$ - it follows that the universal quantification $\forall x\, P(x)$ is the same as the conjuction.

$$P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

because this conjuction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

- When all elements in the domain can be listed - say, $x_1, x_2, \ldots, x_n$ - the existential quantification $\exists x\, P(x)$ is same as the disjunction.

$$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

because this disjunction is true if and only if atleast one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

- **Uniqueness Quantifier:** It is denoted by $\exists!$ or $\exists_1$. The notation $\exists! x P(x)$ [or $\exists_1 x P(x)$] states "There exists a unique x such that $P(x)$ is true".

- **Quantifiers with Restricted Domain:**
An abbreviated notation is often used to restrict the domain of a quantifier. In this notation, a condition that variable must satisfy is included after the quantifier.

Example: What do the statements $\forall x < 0 \, (x^2 > 0)$, $\forall y \neq 0 \, (y^3 \neq 0)$, and $\exists z > 0 \, (z^2 = 2)$ mean, where the domain in each case consists of the real numbers?

Solution: $\forall x < 0 \, (x^2 > 0)$

It states that for every real number x with $x < 0$, $x^2 > 0$. That is, it states "The square of a negative real number is positive". This statement is same as $\forall x \, (x < 0 \rightarrow x^2 > 0)$.

- $\forall y \neq 0 \, (y^3 \neq 0)$
- It states that for every real number y with $y \neq 0$, we have $y^3 \neq 0$. That is "Cube of every non-zero real number is nonzero."
This statement is equivalent to $\forall y \, (y \neq 0 \rightarrow y^3 \neq 0)$.
- $\exists z > 0 \, (z^2 = 2)$
"There is a positive square root of 2".
This statement is equivalent to
$\exists z \, (z > 0 \wedge z^2 = 2)$.

Precedence of Quantifiers:

The quantifiers $\forall$ and $\exists$ have higher precedence then all logical operators from propositional calculus.

For example, $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$. In other words, it means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x (P(x) \vee Q(x))$

Binding Variables: When a quantifier is used on the variable x, we say that this occurrence of the variable is bound.

An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.

The part of a logical expression to which a quantifier is applied is called the scope of this quantifier.

Example $\exists x (x+y=1)$

- Variable x is bound by the existential quantification $\exists x$ but the variable 'y' is free.
- $\exists x (P(x) \wedge Q(x)) \vee \forall x R(x)$

The scope of $\exists x$ is $P(x) \wedge Q(x)$

The scope of $\forall x$ is $R(x)$

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Logical Equivalences Involving Quantifiers.

Definition: Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which ~~what~~ predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.
We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Example: $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$.

Negating Quantifiers.

Negation	Equivalent Statement	When True	When False
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false	There is an x for which $P(x)$ is true.
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false	$P(x)$ is true for all x.

Example: What are the negations of the statements "There is an honest politician" and "All Americans eat cheese burgers"?

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Solution:

Let $H(x)$ denote "x is honest"

Then the statement "There is an honest politician" is represented by $\exists x H(x)$, where the domain contains all the politicians.

The negation of the statement is $\neg \exists x H(x)$, which is equivalent to $\forall x \neg H(x)$

This can be expressed as "Every politician is dishonest."

- Let $C(x)$ denote "x eats cheeseburgers".

Then the statement "All Americans eat cheeseburgers" can be denoted by $\forall x C(x)$.

- The negation of this statement is stated as " ~~Some Am~~ There is an American who does not eat cheese burgers".

It is represented as, $\neg \forall x C(x)$ or $\exists x \neg C(x)$

- Translation From English into Logical Expression:

Example: Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

Solution:

"For every student in this class, that student has studied calculus."

"For every student x, in this class, x has studied calculus."

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$C(x) \rightarrow$ " x has studied Calculus".
$\forall x\, C(x) \rightarrow$ " For every student x in this class, x has studied calculus."

<u>Example:</u> " Every mail larger than one megabyte will be compressed"

→ Let $S(m,y)$ be " mail message m is larger than y megabytes", where the variable has the domain of all mail messages and the variable y is a positive real number, and let $C(m)$ denote " Mail message m will be compressed." Then the specification " Every mail larger than 1 megabyte will be compressed" can be represented by

$$\forall m\, (S(m,1) \rightarrow C(m))$$

- <u>NESTED QUANTIFIERS</u>

Two quantifiers are nested if one is within the scope of the other, such as

$$\forall x\, \exists y\, (x+y=0)$$

- Everything within the scope of a quantifier can be thought of as a propositional function. For Example;

$$\forall x\, \exists y\, (x+y=0)$$

is same thing as $\forall x\, Q(x)$, where $Q(x)$ is $\exists y\, P(x,y)$ where $P(x,y)$ is $x+y=0$

· Example: Assume that the domain for the variables x and y consists of all real numbers. The statement.

$$\forall x \forall y (x+y = y+x)$$

~~Says that for real number x&y, there is a real~~

says that $x+y = y+x$, for all real numbers x & y.

$$\forall x \exists y (x+y=0)$$

For every real number x there is a real number ~~x~~ y and ~~such~~ such that $x+y=0$. This states that every real number has an additive inverse.

Example Translate into English statement

$$\forall x \forall y ((x>0) \wedge (y<0) \rightarrow (xy<0))$$

where the domain for both variables consists of all real numbers.

· Solution: "The product of a positive real number and a negative real number is always a negative real number".

Statement	When True?	When False.
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x, there is a y. for which $P(x,y)$ is false.

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Example: Translate the statement "The sum of two positive integers is always positive" into a logical expressions.

Solution

$$\forall x \forall y ((x>0) \land (y>0) \to (x+y>0))$$

Example: Translate the statement

$$\forall x (C(x) \lor \exists y (C(y) \land F(x,y)))$$

into English, where $C(x)$ is "x has a computer", $F(x,y)$ is "x and y are friends" and the domain for both x and y consists of all students in school.

Solution:

Every student in your school has a computer or has a friend who has a computer.

Example: Express the statement "If a person is female and is a parent, then this persons is someone's mother" as a logical expression involving predicates, quantifiers with a domain consisting of all people and logical connectives.

Solution: For every person x, if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y.

$F(x) \to x$ is female

$P(x) \to x$ is parent

$M(x,y) \to x$ is mother of y.

$$\forall x ((F(x) \land P(x)) \to \exists y M(x,y))$$

$$\forall x \exists y ((F(x) \land P(x)) \to M(x,y))$$

Rules of Inference

Rules of inference can be used as building block to construct more complicated valid argument forms.

Rules of Inference

Rules of Inference	Tautology	Name
p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical Syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive Syllogism

p q $\therefore p \wedge q$	$[(p) \wedge (q)] \rightarrow (p \wedge q)$	Conjuction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution

Example State which rule of inference is the basis of the following argument: "It is below freezing point. Therefore, it is either below freezing or raining now."

Solution: Let p be the proposition "It is below freezing now" and 'q' the proposition "It is raining now". Then the argument is of the form.

$$\frac{p}{\therefore p \vee q}$$

This is an argument that uses addition rule.

Example: "It is below freezing point and raining now. Therefore, it is below freezing now."

p = "It is below freezing now".
q = It is raining

$$\frac{p \wedge q}{\therefore p}$$

<u>Example</u>. If it rains today, the we will not have a barbecue today.

- If we donot have a barbecue today, we will have a barbecue tomorrow.
- Therefore, if it rains today, the we will have a barbecue tomorrow.

$$\begin{array}{l} p \to q \\ q \to r \\ \hline p \to r \end{array}$$

Hypothetical syllogism.

- When there are many premises, several rules of inference are often needed to show that argum-ent is valid.

<u>Example</u>: Show that the hypotheses, "It is not sunny this afternoon and it is colder than Yesterday".

- We will go swimming only if it is sunny.
- If we do not go swimming, then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

leads to conclusion "We will be home by sunset"

<u>Solution</u>:
p = it is sunny this afternoon
q: It is colder than yesterday
r: we will go swimming
s: We will take a canoe trip

t: we will be home by sunset. (o)

Then the hypotheses become.

$\neg p \wedge q$

$r \to p$

$\neg r \to s$

$s \to t$

Step	Reason
1. $\neg p \wedge q$	Hypothesis
2. $\neg p$	Simplification
3. $r \to p$	Hypothesis
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \to s$	Hypothesis
6. s	Modus ponens using 4 & 5
7. $s \to t$	Hypothesis
8. t	Modus ponens using 6 & 7

Example: Show that the hypotheses, "If you send me an e-mail message, then I will finish writing the program". "If you don't send me an e-mail message, then I will go to sleep early" and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed".

Solution.

You send me an e-mail message p

I will finish writing the program q

I will go to sleep early r

I will wake up feeling refreshed s

- The hypotheses are, $p \rightarrow q$
$\neg p \rightarrow r$
$r \rightarrow s$

Conclusion $\neg q \rightarrow s$.

- We need to give valid argument with hypotheses to reach the conclusion.

Step	Reason.
1. $p \rightarrow q$	Hypotheses
2. $\neg q \rightarrow \neg p$	Contrapositive of (1)
3. $\neg p \rightarrow r$	Hypothesis
4. $\neg q \rightarrow r$	Hypothetical syllogism using (2) & (3)
5. $r \rightarrow s$	Hypothesis
6. $\neg q \rightarrow s$	Hypothetical Syllogism using (4) and (5)

Fallacy The fallacy are arguments that are convincing but not true and produce faulty inferences. So fallacy are contingencies rather than tautologies.

The proposition $[(p \to q) \wedge q] \to p$ is not a tautology, because it is false when p is false and q is true.

Example: If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics.
Therefore, you did every problems in this book.

Solution: Let p be the proposition "You did every every problem in this ~~a~~ book".
Let q be the proposition "You learned discrete mathematics".
Then ~~as~~ this argument is of the form:

if $p \to q$
and q
then p.

This is an example of an incorrect argument using the fallacy of affirming the conclusion.

Example: If the conditional statement $p \to q$ is true, and $\neg p$ is true, is it correct to conclude that $\neg q$ is true? In other words, is it correct to assume that you did not learn discrete mathematics if you did not do every problem in the book, assuming that if you do every problem in this book then you will learn discrete mathematics?

RULES of Inference for Quantified Statements.

Universal ~~Insan~~ Instantiation: It is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain, given the premise $\forall x\, P(x)$.

Universal Generalization: It is the rule of inference that states that $\forall x\, P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain. Universal generalization is used when we show that $\forall x\, P(x)$ is true by taking an arbitary element c from the domain and showing that $P(c)$ is true.

~~Ex~~

Existential Instantiation: It is the rule that allows us to conclude that there is an element c in the domain for which P(c) is true if we know that $\exists x P(x)$ is true. We cannot select an arbitrary value c here, but rather it must be a c for which P(c) is true.

Existential Generalization: It is the rule of inference that is used to conclude that $\exists x P(x)$ is true when a particular element c with P(c) true is known.

Rule of Inference	Name
$\forall x P(x)$ / $\therefore P(c)$	Universal instantiation
P(c) for an arbitrary c / $\therefore \forall x P(c)$	Universal generalization
$\exists x P(x)$ / $\therefore$ P(c) for some element c	Existential Instantiation
P(c) for some element c / $\therefore \exists x P(x)$	Existential Generalization

Example: Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science".

Solution:

Let $D(x)$ denote "x is in this discrete mathematics class."

- $C(x)$ denote "x has taken course in computer science."
- Then the premiseas are $\forall x (D(x) \rightarrow C(x))$
  $D(\text{marla})$
- Conclusion $C(\text{marla})$

The following steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\forall x (D(x) \rightarrow C(x))$	Premise
2. $D(\text{marla}) \rightarrow C(\text{marla})$	Universal instantiation
3. $D(\text{marla})$	Premise
4. $C(\text{marla})$	Modus ponens from (2) and (3),

PROOF METHODS

(R)

Basic Terminology:

- **Theorem:** A theorem is a statement that can be shown true.
- **Propositions:** Less important theorems sometimes are called propositions.
- **Proof:** A proof is a valid argument that establishes the truth of a theorem.

The statements used in a proof can include axioms (or postulates), which are statements we assume are true, the premises, if any, of the theorem, and previously proven theorems. Axioms may be stated using primitive terms that do not require definition, but all other terms used in theorems and their proofs must be defined.

- **Lemma:** A less important theorem that is helpful in the proof of other results is called a lemma (plural lemmas or lemmata).

Complicated proofs are usually easier to understand when they are provided using a series of lemmas, where each lemma is proved individually.

- **COROLLARY:** It is a theorem that can be established directly from a theorem that can be proved.

Methods of Proving Theorems.

- To prove a theorem of the form $\forall x (P(x) \to Q(x))$, our goal is to show that $P(c) \to Q(c)$ is true, where c is an arbitrary element of the domain, and then apply universal generalization. In this proof, we need to show that a conditional statement is true. Because of this, we focus on methods that show that conditional statements are true. $p \to q$ is true unless p is true and q is false. When the statement $p \to q$ is proved, it need only be shown that, q is true if p is true.

(A) <u>Direct Proofs</u>: A direct proof shows that a conditional statement $p \to q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs.

In a direct proof, we assume that p is true and use axioms, definitions and previously proven theorems, together with rules of inference, to show that q must also be true.

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<u>Definition 1</u> : The integer n is even if there exists an integer k such that $n = 2k$, and n is odd if there exists an integer k such that $n = 2k+1$. (Note that an ~~either~~ integer is either even or odd, and no integer is both even and odd).

<u>Example 1</u> : Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

<u>Solution</u>: Theorem states that $\forall n \, (P(n) \to Q(n))$

P(n) is " n is an odd integer"

Q(n) is " n^2 is odd"

- To begin a direct proof, we assume that the hypothesis of this conditional statement is true, namely we assume that n is odd.
- By definition of odd integer,

$n = 2k+1$, where k is some integer

Squaring both sides we get

$$n^2 = (2k+1)^2$$

$$\text{or } n^2 = 4k^2 + 4k + 1$$

$$\text{or } n^2 = 2(2k^2 + 2k) + 1$$

By the definition of an odd integer, we conclude that n^2 is an odd integer (it is one more than twice an integer).

Consequently, we have proved that if n is an odd integer, then n^2 is an odd integer.

Example 2: Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square. (An integer 'a' is a perfect square if there is an integer 'b' such that $a=b^2$).

Solution:

We assume that m and n are both perfect squares.

By definition of perfect squares

$$m = s^2 \quad \text{---(i)}$$
$$n = t^2 \quad \text{---(ii)}$$
[s and t are integers]

Multiplying (i) & (ii)

$$m \cdot n = s^2 \cdot t^2$$

or $$m \cdot n = (s \cdot t)^2$$

- By the definition of perfect square, it follows that mn is also a perfect square, because it is the square of $s \cdot t$, which is an integer.

B) Proof by Contraposition

- Proofs by contraposition make use of fact that the conditional statement $p \to q$ is equivalent to its contrapositive, $\neg q \to \neg p$. This means that the conditional statement $p \to q$ can be ~~st~~ proved showing that its contrapositive, $\neg q \to \neg p$ is true. In a proof by contraposition of $p \to q$, we take $\neg q$ as hypothesis and using axioms, definitions and previously prove theorems, together

- with rules of inference, we show that $\neg p$ must follow.

Proofs, that do not start with the hypothesis and end with the conclusion, are called <u>indirect proofs</u>.

<u>Example:</u> Prove that if n is an integer and $3n+2$ is odd, then n is odd.

<u>Solution:</u>

Direct Proof

We assume that $3n+2$ is odd

$\therefore$ $3n+2 = 2k+1$, for some integer k

$3n+1 = 2k$

But this does not conclude that n is odd.

- Proof by Contraposition.

The first step here is to assume that the conclusion of the conditional ~~stament~~ statement "If $3n+2$ is odd, then n is odd" is false; namely.

assume n is even

$\therefore$ $n = 2k$ for some integer k.

Substituting $2k$, we find that

$3n+2 = 3(2k)+2 = 6k+2$

$= 2(3k+1)$

Vacuous and Trivial Proofs

We can quickly prove that a conditional statement $p \to q$ is true when we know that p is false, because $p \to q$ must be true when p is false. Consequently, if we can show that p is false, then we have a proof, called a vacous proof, of the conditional statement $p \to q$.

Example: Show that the proposition $P(0)$ is true, where $P(n)$ is "if $n > 1$, then $n^2 > n$" and the domain consists of all integers.

Solution: Note that $P(0)$ is "If $0 > 1$, then $0^2 > 0$". We can show $P(0)$ using a vacuous proof, because the hypothesis $0 > 1$ is false. This tells us that $P(0)$ is automatically true.

- We can also prove a conditional statement $p \to q$ if we know that the conclusion q is true. By showing q is true, it follows that $p \to q$ is also be true. A proof of $p \to q$ that uses the fact that q is true is called trivial proof.

Example Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$", where the domain consists of all integers. Show that $P(0)$ is true.

Solution: The proposition $P(0)$ is "If $a \geq b$, then $a^0 \geq b^0$". Because $a^0 = b^0 = 1$, the conclusion of the conditional statement "If $a \geq b$, then $a^0 \geq b^0$" is true. Hence, this conditional, which is $P(0)$, is true. This is an example of a trivial proof. The hypothesis, which is the statement "$a \geq b$", was not needed in this proof.

Example: Prove that if n is an integer and n^2 is odd, then n is odd.

Solution: We first attempt a direct proof.
Suppose that n is an integer and n^2 is odd. Then, there exist an integer k, such that

$$n^2 = 2k+1.$$

Can we use this information to show, n is odd?

$n = \pm\sqrt{2k+1}$ → This is not useful.

- Proof by contraposition.

p is "if n is an integer and n^2 is odd"

q is "n is odd"

$\neg q$ is "n is not odd" i.e "n is even"

$\neg p$ is "n^2 is even (not odd)".

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This implies that there exists an integer k such that

$$n = 2k$$

To prove the theorem, we need to ~~prove~~ show that this hypothesis implies the conclusion that n^2 is not odd, that n^2 is even.

$$n = 2k$$

Squaring both sides we obtain

$$n^2 = (2k)^2$$

$$n^2 = 4k^2$$

$$n^2 = 2(2k^2)$$

which implies that n^2 is also even as

$$n^2 = 2t \quad (\text{where } t = 2k^2)$$

- We have proved that if n is an integer and n^2 is odd, then n is odd.

<u>Proof by Contradiction.</u>

Suppose that we want to prove that a stateme-nt p is true. Furthermore, suppose that we can find a contradiction q such that $\neg p \to q$ is true. Because q is false, but $\neg p \to q$ is true, we can conclude that $\neg p$ is false, which means that p is true.

Because the statement $r \wedge \neg r$ is a contradiction whenever r is a proposition, we can prove that p is true if we can show that $\neg p \to (r \wedge \neg r)$ is true for —

- some proposition r. This type of proofs are called <u>proofs by contradiction.</u>

Example: ~~Proof~~ Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution: Let p be the proposition "$\sqrt{2}$ is irrational". We suppose that

$\neg p$ is true.

$\neg p$ is the statement "It is not the case that $\sqrt{2}$ is irrational," which says that "$\sqrt{2}$ is rational."

We will show that assuming that $\neg p$ is true leads to a contradiction.

If $\sqrt{2}$ is rational, there exists integers a and b with

$\sqrt{2} = \frac{a}{b}$ (where a and b have no common factors so that a fraction a/b is in lowest terms.)

Here, we are using the fact that every rational number can be written in lowest terms.

$$\sqrt{2} = \frac{a}{b}$$

Squaring both sides, we get

$$2 = \frac{a^2}{b^2}$$

By the definition of an even integer it follows that a^2 is even. Also, if a^2 is even, a must also be even.

$\therefore$ Let $a = 2c$ for some integer c. Thus

$$2b^2 = 4c^2$$

Dividing both sides of this equation by 2 gives.

$$b^2 = 2c^2$$

By the definition of even, this means b^2 is even. Again using the fact that if the square of an integer is even, then integer itself must be even.

- Since a and b both ~~be~~ are even, both ~~b~~ are divided by 2, but we have assumed that ~~Op~~ $\sqrt{2} = \frac{a}{b}$, where a and b have no common factors.

- Because our assumption of $\neg p$ leads to the contradiction, $\neg p$ must be false, i.e p must be true.

That is "$\sqrt{2}$ is ~~r~~irrational", this statement P is true.

(W)

Proof by counter examples

To show that a statement of the form $\forall x P(x)$ is false, we need ~~...~~ only find a counterexample, that is, an example x for which $P(x)$ is false. When presented with a statement of the form $\forall x P(x)$, which we believe to be false or which has resisted all proof attempts, we look for a counter example.

Example: Show that the statement "Every positive integer is the sum of the squares of two integers".

Solution: To show that this statement is false, we look for a counter example, which is a particular integer that is not the sum of the squares of two integers.

3 cannot be written as the sum of the squares of two integers. The only ~~or~~ perfect squares not exceeding 3 are $0^2 = 0$ and $1^2 = 1$. Furthermore, there is no way to get 3 as sum of two terms each of which is 0 or 1.

Consequently, we have shown that "Every positive integer is the sum of the squares of two integers" is false.

Mistakes in Proofs

Example: What is wrong ~~with~~ with this famous supposed "proof" that 1=2?

"Proof:" We use these steps, where a and b are two equal positive integers.

Step	Reason
1. $a=b$	Given
2. $a^2=ab$	Multiply both sides of (1) by a
3. $a^2-b^2=ab-b^2$	Subtract b^2 from both sides of 2.
4. $(a-b)(a+b)=b(a-b)$	Factor both sides of (3)
5. $a+b=b$	Divide both sides of (4) by $a-b$
6. $2b=b$	Replace a by b in (5) because $a=b$ and simplify
7. $2=1$	Divide both sides of (6) by b.

Solution: Every step is valid except for one, step 5 where we divide both sides by $a-b$. The error is that $a-b$ equals zero; division of both sides of an equation by the same quantity is valid as long as this quantity is not zero.

- Many incorrect arguments are based on a fallacy called begging the question. This fallacy occurs when one or more steps of a proof are based on the truth of the statement being proved. In other words, this fallacy arises when a statement is proved using itself, or a statement equivalent to it. That is why this fallacy is also called circular ~~[illegible]~~ reasoning.

Example 18: Is the following argument correct? It supposedly shows that n is an even integer whenever n^2 is an even integer.

Suppose that n^2 is even. Then $n^2 = 2k$ for some integer k. Let $n = 2l$ for some integer l. This shows that n is even.

Solution: This argument is incorrect.
The statement "let $n = 2l$ for some integer l" occurs in the proof. No argument has been given to show that n can be written as $2l$ for some integer l. This is a circular reasoning because this statement is equivalent being proved, namely, "n is even".

Exhaustive Proof and Proof by Cases

To prove a conditional statement of the form:

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$$

the tautology

$$[(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)]$$

can be used as rule of inference. This show that the original conditional statement with a hypothesis made up of a disjunction of the propositions $p_1, p_2, \dots p_n$ can be proved by proving each of the n conditional statements $p_i \to q$, $i = 1, 2, \dots n$ individually. Such an argument is called a proof by cases.

Exhaustive Proof: Some theorems can be proved by examining a relatively small number of examples. Such proofs are called exhaustive proofs. An exhaustive proof is a special type of proof by cases where each case involves checking a single example.

Example (Exhaustive Proof). ~~Prove that $(n+1)^2 \geq 3^n$~~ if n is a positive integer with $n \leq 4$.

Solution: We use a proof by exhaustion.

for $n=1$	for $n=2$	for $n=3$	for
$(1+1)^2 \geq 3^1$	$(2+1)^2 \geq 3^2$	$16 \geq 9$	
$4 \geq 3$	$9 \geq 9$		

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Example (Proof by Exhaustion)

if $2 \le n \le 5$; $n \in \mathbb{N}$; then $4 \nmid n^2+2$

(does not divide)

Solution

Case I: $n=2$. $\quad n=3 \quad n=4 \quad n=5$

$4 \nmid 6$. $\quad 4 \nmid 11 \quad 4 \nmid 18 \quad 4 \nmid 27$

Example if $n \in \mathbb{Z}$ then $4 \nmid n^2+2$. (Proof by cases)

Solution

Case I: n is even

$n = 2m$, $m \in \mathbb{Z}$

$n^2+2 = (2m)^2+2$

$n^2+2 = 4m^2+2$

or $4m^2+2 = 4k$ $\quad k \in \mathbb{Z}$ ($\because$ assuming 4 divides $4m^2+2$)

Dividing both sides by 2 we get

$2m^2+1 = 2k$

if k is an integer then $2k$ is even

if m is an integer then $2m^2+1$ is odd

but odd $\neq$ even

$\therefore$ n^2+2 is not divided by 4.

Case II n is odd

$\therefore$ $n = 2m+1$ $\quad m \in \mathbb{Z}$

$(2m+1)^2+2 = 4m^2+4m+1+2$

$= 2(2m^2+2m+1)+1$

$2(2m^2+2m+1)+1 = 4k$ [$k \in \mathbb{Z}$, assuming LHS is divided by 4]

But, we have odd term on LHS and even term on RHS.

$\therefore$ $4 \nmid n^2+2$

when n is odd.

<u>Example.</u> Prove if $n \in \mathbb{Z}$, n^2+3n+4 is even.

<u>Case I</u> n is odd.

$n = 2k+1$

$$n^2+3n+4 = (2k+1)^2 + 3(2k+1) + 4$$
$$= 4k^2 + 10k + 12$$
$$= 2(2k^2+5k+6) \quad \because \text{RHS is multiple of 2.}$$

$\therefore$ n^2+3n+4 is even

<u>Case II</u> n is even

$n = 2k$

$$n^2+3n+4 = (2k)^2 + 3(2k) + 4$$
$$= 4k^2 + 6k + 4$$
$$= 2(2k^2+3k+2) \rightarrow \text{is multiple of 2.}$$