UNIT: 6

Relations and Graphs.

6.1: Relations

Definition: Let A and B be the two non-empty sets. A relation from A to B is any subset of the Cartesian product A×B satisfying given specific condition.

i.e $R \subseteq A \times B$

Suppose R is a relation from A to B. Then R is a set of ordered pairs (a,b) where $a \in A$ and $b \in B$. Every ordered pair (a,b) is written as aRb, read as 'a is related to b by R'. If $(a,b) \notin R$ then 'a' is not related to 'b' by R and is written $a \not R b$.

If R is a relation from a set A to itself, that is, if R is subset of $A^2 = A \times A$, then we say R is relation on A.

Domain and Range

Example: Let $A = \{4, 5, 6\}$, find the relations in $A \times A$ under the condition $x+y < 10$. Also find domain and range of relation.

Solution:

$$A \times A = \{(4,4), (4,5), (4,6), (5,4), (5,5), (5,6), (6,4), (6,5), (6,6)\}$$

The given condition is : $x+y<10$

So, $R = \{(4,4), (4,5), (5,4)\}$

$Dom(R) = \{4,5\}$ $Range(R) = \{4,5\}$

Properties of Relation:

Reflexive Relation: A relation R on a set is reflexive if $(a,a) \in R$ for every element $a \in A$.

Example:

(a) If $R = \{(1,1), (1,2), (2,2), (2,3), (3,3)\}$ is a relation on $A = \{1,2,3\}$, then R is reflexive relation since every $a \in A$, $(a,a) \in R$.

(b) If $R = \{(1,1), (1,2)\}$ is a relation on $B = \{1,2,3\}$. Then R is irreflexive since for $2 \in B$ there is no $(2,2)$ in R and for $3 \in B$ there is no $(3,3) \in R$.

(c) $R = \{(x,y) \in R^2 : x \leq y\}$ is a reflexive relation since $x \leq x$ for any $x \in R$. (a set of Real numbers).

(d) $S = \{(x,y) \in R^2 : x < y\}$ is an irreflexive relation since $x < x$ for no $x \in R$. (the set or real numbers).

Symmetric Relation: A relation R on a set A is symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$.

Asymmetric Relation: A relation R on a set A is is asymmetric if $(a,b) \in R$ ~~whenever~~ then $(b,a) \notin R$ for all $a, b \in A$.

Antisymmetric Relation: A a relation R on a set A is antisymmetric if $a=b$ whenever aRb and bRa. The contrapositive of this definition is that R is antisymmetric if $a \not R b$ or $b \not R a$ whenever $a \neq b$.

Example:

(a) $R_1 = \{(1,1), (1,2), (1,3), (2,3), (2,1)\ (3,2), (3,1)\}$

is symmetric relation since for $(1,2), (1,3)\ (2,3)$ there are $(2,1)$, $(3,1)$ and $(3,2)$ respectively.

(b) $S = \{(x,y) \in R^2 : x^2+y^2=1\}$ is symmetric relation since $y^2+x^2=1$. ~~so~~ clearly R_2 contains (y,x) which satisfies $y^2+x^2=1$.

(c) $S = \{(1,1), (1,2), (2,3), (3,1)\}$ on $A = \{1,2,3\}$ is asymmetric since for $(1,2) \in S$, there is no $(2,1)$ in S.

(d) $R = \{(1,2), (2,2), (2,3)\}$ on $A = \{1,2,3\}$ is an antisymmetric since if we choose 1 and 2 then for $1 \neq 2$, $(1,2) \in R$ but $(2,1) \in R$. Again if we choose 2 and 3, ~~for 9~~ then for $2 \neq 3$, $(2,3) \in R$ but $(3,2) \in R$.

Transitive Relation:

A relation R on a set A is transitive if whenever aRb and bRc, then aRc i.e $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$ for all $a, b, c \in A$

Example

(a) Let $A = \{1,2,3\}$ and $R = \{(1,2), (3,2), (2,3), (1,3), (2,2), (3,3)\}$ the R is transitive.

(b) Let $A = \{1,2,3,4\}$ and $R = \{(1,2), (1,3), (4,2)\}$ then R is intransitive since there are no elements a, b and c in A such that aRb and bRc, but $a \not{R} c$.

Combining Relations:

The relation from set A to B subsets of A×B so two relations from A to B can be Combined in a same way that two sets can be Combined.

Example:

Let $A = \{4,5,6\}$ and $B = \{4,5,6,7\}$. The relations $R_1 = \{(4,4),(5,5),(6,6)\}$ and $R_2 = \{(4,4),(4,5),(4,6),(4,7)\}$ then find $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$ and $R_2 - R_1$

Solution:

$R_1 \cup R_2 = \{(4,4),(5,5),(6,6),(4,5),(4,6),(4,7)\}$

$R_1 \cap R = \{(4,4)\}$

$R_1 - R_2 = \{(5,5),(6,6)\}$

$R_2 - R_1 = \{(4,5),(4,6);(4,7)\}$

Types of Relation:

Complementary Relation: Let R be a relation from a set A to B. The complementary relation of R is denoted by $\overline{R}$ which consists of those ordered pairs which are not in R, that is.

$\overline{R} = \{(a,b) \in A \times B : (a,b) \notin R\}$

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Example: Let R be relation on set $A = \{1, 2, 6\}$ defined as $R = \{(x, y) : x < y\}$. Find the complement relation of R.

Solution:

$A \times A = \{(1,1), (1,2), (1,6), (2,1), (2,2), (2,6), (6,1), (6,2), (6,6)\}$

$R = \{(1,2), (1,6), (2,6)\}$

$\overline{R} = \{(1,1), (2,1), (2,2), (6,1), (6,2), (6,6)\}$

Inverse Relation:

Let R be a relation from A to B. The inverse of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs which when reversed, belongs to R; that is

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Example:

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$.

Solution: Find R^{-1} if $R = \{(1,a), (1,b), (2,b), (2,c), (3,b)\}$

$R^{-1} = \{(a,1), (b,1), (b,2), (c,2), (b,3)\}$

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Identity Relation

A relation R in a set A i.e. a relation R from A to A is said to be a identity relation, generally denoted by I_A, if

$$I_A = \{(x,x) : x \in A\}$$

Example

Let $A = \{1,2,3\}$ then $I_A = \{(1,1), (2,2), (3,3)\}$

N-ary Relations:

Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the domains of the relation, and n is called its degree.

Example: Let $A = \{1,2\}$ and let R be the relation defined by the property '$x+y+z$ is even'

Now, $A \times A \times A = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)\}$

Under '$x+y+z$ is even' the relation is

$R = \{(1,1,2), (1,2,1), (2,1,1), (2,2,2)\}$ which is a ternary relation.

Operations on N-ary Relation:

There are a variety of operations on N-ary relations that can be used to form new n-ary relations. The most common operation is determining all N-tuples in the N-ary relation that satisfy certain conditions.

1. Let R be an n-ary relation and C a condition that elements in R may satisfy. Then the selection operator S_c maps the N-ary ~~relation R to the N-ary relation of all n-tuples from R that satisfy certain~~ relation R to the N-ary relation of all n-tuples from R that satisfy the condition C.

2. The projection $P_{i_1, i_2, \dots, i_n}$ maps the N-tuple $(a_1, a_2, \dots, a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, \dots, a_{i_m})$ where $m \le n$.

3. Let R be a relation of degree m ~~of~~ and S a relation of degree n. The join $J_p(R,S)$, where $p \le m$ and $p \le n$, is a relation of degree $m+n-p$ that consists of all $(m+n-p)$ tuples $(a_1, a_2, \dots a_{m-p}, c_1, c_2, \dots, c_p, b_1, b_2, \dots, b_{n-p})$ where the ~~m~~ m-tuple $(a_1, a_2, \dots, a_{m-p}, c_1, c_2, \dots c_p)$ belongs to R and the n-tuple $(c_1, c_2, \dots, c_p, b_1, b_2, \dots, b_{n-p})$ belongs to S.

Example: Let R be the relation on $Z \times Z \times Z^+$ consisting of triples (a, b, m), where a, b and m are integers with $m \geq 1$ and $a \equiv b \pmod{m}$.

Then, $(8, 2, 3)$, $(-1, 9, 5)$ and $(14, 0, 7)$ all belong to R but

$(7, 28, 3)$, $(-2, -8, 5)$ and $(11, 0, 6)$ do not belong to R.

This relation has degree 3 and its first two domain are the set of all integers and its third domain is the set of positive integers.

Example of Operation I

Table I

Student	ID number	Major	GPA
Ackermann	23456	Computer Science	3.38
Adams	12567	Physics	3.48
Chou	13567	Computer Science	3.49
Goodfri	49378	Mathematics	2.03
Rao	13978	Psychology	5.00

- To find the records of Computer science majors in the n-ary relation R shown is table above, we use the operator S_{C_1}, where C_1 is the condition Major = "Computr Sciene". The result is the two 4-tuples

(Ackermann, 23456, Computer science, 3.38)

& (Chou, 13567, Computer science, 3.49)

Example of operation II: What ~~is~~ relation results when the projection $P_{1,4}$ is applied to the relation in the table 1?

Solution:

Student	GPA
Ackermann	3.38
Adams	3.48
Chou	3.49
Goodfriend	2.03
Rao	5.00

Example of operation III:

Prof	Department	Course No.
Cruz	Zoology	416
Cruz	Zoology	420
Farber	Physics	516
Farber	Physics	530
Rosen	Mathematic	319

Department	~~Room~~ Time	~~Time~~ Room	Course No.
Zoology	7:00	412A	416
Physics	8:00	512D	530
Botany	9:00	600C	213
Mathematics	10:00	516B	319
Zoology	8:00	412A	420

Prof	Department	Course No.	Time	Room
Cruz	Zoology	416	7:00	412 A
Cruz	Zoology	420	8:00	412 A
Fraber	Physics	530	8:00	512 D
Rosen	mathematics	319	10:00	516 B

Example:

Table: flight.

Airline	Flight no	Gate	Destination	Departure time
Nadir	123	24	Detroit	8:10
Acme	456	46	Berlin	2:10
Acme	789	68	Detroit	8:30
Nadir	101	89	Detroit	9:30
Acme	112	42	Rome	10:30

. The SQL statement

```
SELECT Departure-time
FROM flight
WHERE Destination = 'Detroit'
```

The output of above selection operation results the following table.

Departure_time
8:10
8:30
9:30

SQL uses the FROM clause to identify the n-ary relation the query is applied to, the WHERE clause to specify the condition of the selection operation, and the SELECT clause to specify the projection operation that is applied.

COMPOSITION OF RELATION

Let A, B, C be three sets. Let R be a relation from A to B and S be a relation from B to C. Then the composite of R and S is denoted by SOR and defined as

$$SOR = \{(a,c) \in A\times C : \text{for some } b \in B,\ (a,b) \in R \text{ and } (b,c) \in S\}$$

That is, $a(SOR)c$, if for some $b \in B$, we have aRb and bSc.

Example: Let $A = \{1,2,3\}$, $B = \{p,q,r\}$, $C = \{x,y,z\}$ and let $R = \{(1,p),(1,r),(2,q),(3,q)\}$ and

$S = \{(p,y),(q,x),(r,z)\}$

Compute SOR.

Solution:

R is ~~tea~~ relation from A to B and S is relation from B to C.

The ordered pairs $(1,p) \in R$ and $(p,y) \in S$ produce the order pair $(1,y) \in S \circ R$, for some $p \in B$

Similarly, $(1,z) \in S \circ R$ for $r \in B$
$(2,x) \in S \circ R$ for $q \in B$.
$(3,x) \in S \circ R$ for $q \in B$

$\therefore\ S \circ R = \{(1,y), (1,z), (2,x), (3,x)\}$

REPRESENTING RELATIONS:

↳ USING MATRICES

↳ Appropriate for the ~~representing~~ representation of relations in computer programs.

↳ USING DIRECTED GRAPHS

↳ Useful for understanding the property of the relation.

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Representing Relations Using Matrices:

- A relation between finite sets can be represented using a zero-one matrix.

Suppose that R is a relation from $A = \{a_1, a_2 \ldots a_n\}$ to $B = \{b_1, b_2, \ldots, b_n\}$. The relation R can be represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

- In other words, the zero-one matrix representing R has a '1' as its (i, j) entry when a_i is related to b_j and a '0' in this position if a_i is not related to b_j.

Example: Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$ and $a > b$. What is the matrix representating R if $a_1 = 1$, $a_2 = 2$, & $a_3 = 3$, & $b_1 = 1$, $b_2 = 2$

Solution: Because $R = \{(2,1), (3,1), (3,2)\}$ the matrix for R is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The 1s in M_R show that pairs $(2,1)$, $(3,1)$ & $(3,2)$ belongs to R.

Example: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$ which ordered pairs are in the relation R represented by the matrix.

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution:

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$$

- The matrix of a relation on a set, which is a square matrix can be used to determine whether the relation has certain properties.
- A Relation R is reflexive if $(a, a) \in R$ whenever $a \in A$. Thus, R is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots, n$. Hence, R is reflexive if and only if $m_{ii} = 1$, for $i = 1, 2, \ldots n$. In other words, R is reflexive if all the ~~relation~~ elements on the main diagonal of M_R are equal to 1.

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

Fig: The zero-one matrix for a Reflexive relation.

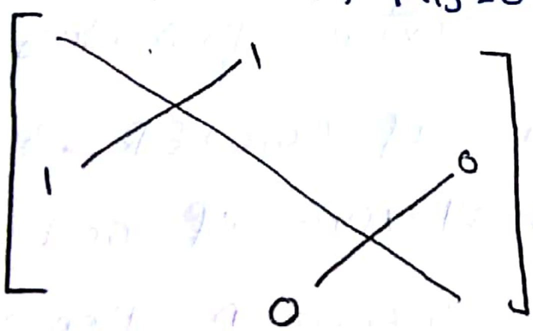
The relation R is symmetric if $(a,b) \in R$ implies that $(b,a) \in R$. In terms of the enteries of M_R, R is symmetric if and only if $m_{ji} = 1$ whenever $m_{ij} = 1$. This also means $m_{ji} = 0$ whenever $m_{ij} = 0$.

R is symmetric if and only if

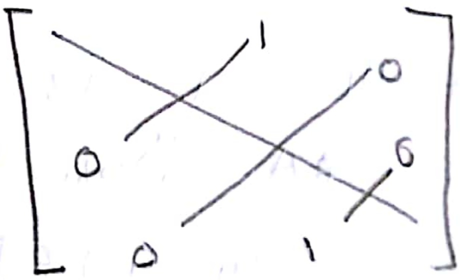
$$M_R = (M_R)^t$$

that is, if M_R is a symmetric matrix.

- The relation R is antisymmetric if and only if $(a,b) \in R$ and $(b,a) \in R$ imply that $a = b$. The matrix of antisymmetric relation has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. Or in other words, either, $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



(a) Symmetric



(b) Antisymmetric

Example: Suppose that the relation R on a set is represented by the matrix.

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric ~~or~~ and/or antisymmetric?

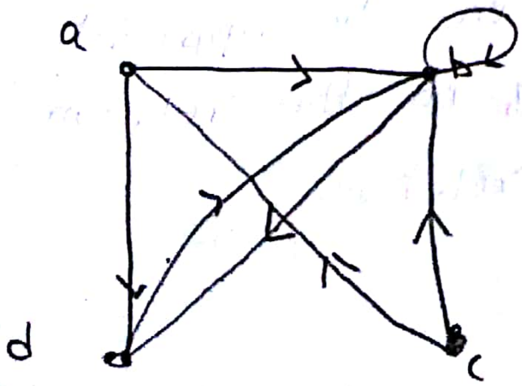
~~Representations~~

Representing Relations Using Digraphs.

It is Pictorial representation.

Each element of the set is represented by a Point and each ordered is represented using an arc with its direction indicated by an arrow.

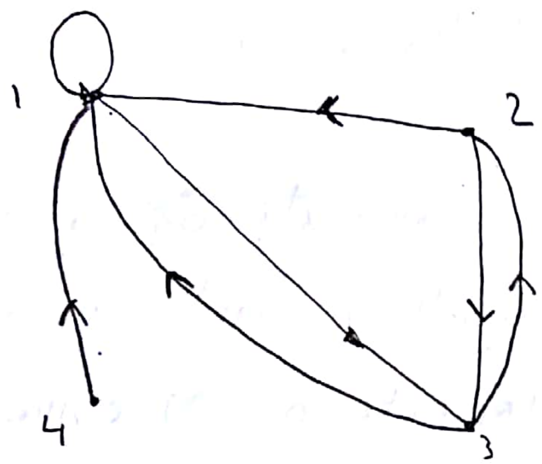
- A directed graph or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex 'a' is called the initial vertex of the edge (a,b) and the vertex b is called the terminal vertex of this edge.
- An edge of the form (a,a) is represented using an arc from the vertex a back to itself. Such an edge is called a loop.
- **Example:** The directed graph with vertices a, b, c and d and edges (a,b), (a,d), (b,b), (b,d), (c,a), (c,b) and (d,b) is displayed in following figure.



Example: The directed graph of the relation.

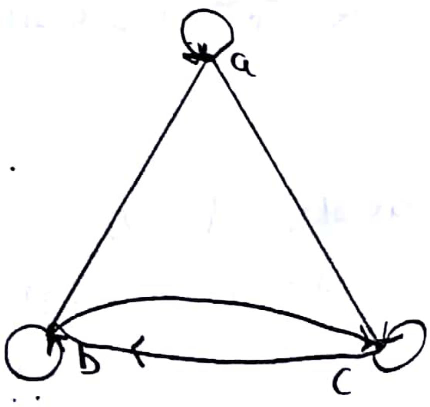
$R = \{(1,1), (1,3), (2,1), (2,3), (3,1), (3,2), (4,1)\}$

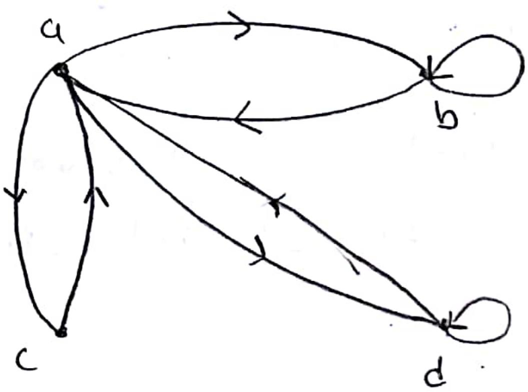
on the set $\{1,2,3,4\}$ is shown in the following figure.



- A relation is reflexive if and ~~if~~ only if there is a loop at every vertex of the directed graph, so that the every ordered pair of the form (x,x) occurs in the relation.
- A relation is symmetric if and only if for every edge between distinct vertices in its diagraph there is an edge in the opposite direction, so that (y,x) is in the relation whenever (x,y) is in the relation.

- A relation is antisymmetric if and only if there are never two edges in opposite directions between distinct vertices.
- A relation is transitive if and only if whenever there is an edge from a vertex x to a vertex y and an edge from a vertex y to a vertex z, there is an edge from a vertex x to z.





CLOSURES OF RELATIONS:

Reflexive closure: The reflexive closure of a binary relation R on a set X is the smallest reflexive relation on X that contains R.

Eg. If X is the set of distinct numbers and xRy means "x is less than y", then the reflexive closure of R is the relation "x is less than or equal to y".

eg. The relation $R = \{(1,1), (1,2), (2,1), (3,2)\}$ on the set $A = \{1,2,3\}$ is not reflexive.

- We can add $(2,2)$ and $(3,3)$ to R to make it reflexive, because these are the only pairs of the form (a,a) that are not in R.
  This new relation contains R. Any reflexive relation that contains R must also contain $(2,2)$ and $(3,3)$. Because this relation contains R, is reflexive and is contained within every reflexive relation that contains R, it is called the reflexive closure of R.

∴ The reflexive closure of R equals, $R \cup \Delta$, where $\Delta = \{(a,a) \mid a \in A\}$ is the diagonal relation on A.

SYMMETRIC CLOSURE:

The symmetric closure of a relation R on a set X is the smallest symmetric relation on X that contains R.

The relation $\{(1,1), (1,2), (2,2), (2,3), (3,1), (3,2)\}$ on $\{1,2,3\}$ is not symmetric.
When we add $(2,1)$ and $(1,3)$ to this relation it will be symmetric, because these are the only pairs of the form (b,a), with $(a,b) \in R$ that are not in R.

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The symmetric closure of a ~~sets~~ relation can be constructed by taking a union with its inverse that is, $R \cup R^{-1}$ is the symmetric closure of R, where, $R^{-1} = \{(b,a) \mid (a,b) \in R\}$.

Transitive Closure:

Transitive closure of a relation R on a set X is the smallest relation on X that contains R and is transitive.

For example: If X is a set of airports and xRy means "there is a direct flight from airport x to airport y" (for x and y in X), then the transitive closure of R on X is the relation R^+ such that xR^+y means "it is possible to fly from x to y in one or more flights"

Equivalence Relation:

A relation R on a set A is called an equivalence relation if ~~it~~ is reflexive, symmetrical and transitive.

Example:

Let $A = \{1, 2, 3, 4, 5\}$

Show that the relation

$R = \{(1,1), (1,5), (2,2), (2,4), (3,3), (4,2), (4,4), (5,1), (5,5)\}$.

Solution:

Given, $A = \{1, 2, 3, 4, 5\}$

$R = \{(1,1), (1,5), (2,2), (2,4), (3,3), (4,2), (4,4), (5,1), (5,5)\}$

i) Reflexive: $\forall a \in A, (a,a) \in R$

$(1,1) \in R, (2,2) \in R, (3,3) \in R, (4,4) \in R, (5,5) \in R$

$\therefore$ R is ~~B~~ Reflexive

ii) Symmetric:

$1, 5 \in A, (1,5) \in R \Rightarrow (5,1) \in R$

$2, 4 \in A, (2,4) \in R \Rightarrow (4,2) \in R$

$\therefore$ R is symmetric

iii) Transitive: $\forall a, b, c \in A, (a,b) \in R \cap (b,c) \in R \Rightarrow (a,c) \in R$

$(1,5) \in R \wedge (5,1) \in R \Rightarrow (1,1) \in R$

$(2,4) \in R \wedge (4,2) \in R \Rightarrow (2,2) \in R$

$(2,4) \in R \wedge (4,4) \in R \Rightarrow (2,4) \in R$

$(5,1) \in R \wedge (1,5) \in R \Rightarrow (5,5) \in R$

$\therefore$ R is transitive.

Since, R ~~is~~ satisfies reflexive, symmetric and transitive behaviour.

Example: Consider the following relation on $\{1, 2, 3, 4, 5, 6\}$ $R = \{(i,j) : |i-j| = 2\}$

Is R reflexive? Is R symmetric? Is R transitive?

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~~Let a and b be two~~

Solution:

Let $A = \{1,2,3,4,5,6\}$. Then

$R = \{(i,j) : |i-j| = 2\}$ on A

$= \{(1,3), (2,4), (3,1), (4,2), (3,5), (5,3), (4,6), (6,4)\}$

not reflexive, not transitive but symmetric

$\therefore$ not equivalence relation.

Congurence Modulo Relation:

Let a and b be two integers then 'a' is congurence to 'b modulo m' if m divides $a-b$

$a \equiv b \pmod m$

Example: Let m be a positive integer with $m>1$ show that the relation $R = \{(a,b) : a \equiv b \pmod m\}$ is an equivalence on a set of positive integers.

Solution:

i) Reflexive, $\forall\ a \in Z^+$

$a \equiv a \pmod m \Rightarrow m \mid (a-b) \Rightarrow m \mid 0$ (true)

$\therefore (a,a) \in R$, R is reflexive.

ii) Symmetric, $\forall\ a, b \in Z^+$

Let $(a,b) \in R$

So, $a \equiv b \pmod m \Rightarrow m \mid (a-b)$

$a-b = m \times k$

or $b-a = m \times (-k)$ or $b \equiv a \pmod m$

$m \mid (b-a)$ $\therefore (b,a) \in R$, Hence, R is symmetric

(iii) Transitive: $\forall a,b,c \in Z^+$

let, $(a,b) \in R$

$\Rightarrow a \equiv b \pmod{m} \Rightarrow m \mid (a-b)$

$a-b = m \times k_1$ — (i)

Again $(b,c) \in R$

$\Rightarrow b \equiv c \pmod{m}$

$m \mid b-c$

$b-c = m \times k_2$ — (ii)

adding (i) and (ii) we get

$a-b+b-c = m * k_1 + mk_2$

$a-c = m(k_1+k_2)$

$\therefore m \mid a-c$

$\Rightarrow a \equiv c \pmod{m}$

$\therefore (a,c) \in R$, $\therefore$ R is transitive.

Hence, the given relation R is an equivalence relation.

Equivalence Classes:

If R is an equivalence relation on a set A and xRy then x and y are called equivalent with respect to R. Then the class of any element $x \in A$ is denoted by $[x]$ which is defined as

$$[x]_R = \{y \in A : (x,y) \in R\}$$

The collection of all equivalence classes of elements under an equivalence relation R is denoted by A/R, that is

$$A/R = \{[x] : x \in A\}$$

It is called the quotient set of A by R.

Example: Let $R = \{(1,2), (2,1), (1,1), (2,2), (3,3), (4,4)\}$ be a relation on $A = \{1,2,3,4\}$. Find the equivalence classes of each element of A and quotient set of A by R.

Solution.

Equivalence Classes.

$[1]_R = \{1, 2\}$

$[2]_R = \{1, 2\}$

$[3]_R = \{3\}$

$[4]_R = \{4\}$

And $A/R = \{\{1,2\}, \{3\}, \{4\}\}$

- Equivalence classes are either equal or disjoint

  i.e either $[a]_R = [b]_R$

  OR

  if $[a]_R \neq [b]_R$

  $[a]_R \cap [b]_R = \emptyset$

- Also $\bigcup_{a \in A} [a]_R = A$

- In above example,

  $[1]_R = [2]_R$, $[2]_R \cap [3]_R = \emptyset$

  $[4]_R = [3]_R$

  $[1]_R \cup [2]_R \cup [3]_R \cup [4]_R = A$

Partition of Sets

A partition set or quotient set, of a set A is the collection of subsets of A i.e

$P = \{A_1, A_2, \dots, A_n\}$ such that

(i) Union of A_i is A

(ii) For distinct A_i and A_j, $A_i \cap A_j = \emptyset$

The sets in P are called blocks or cells of partition.

Example:

Let $A = \{a, b, c, d, e, f, g, h\}$ and $P = \{A_1, A_2, A_3, A_4, A_5\}$ where

$A_1 = \{a, b, c, d\}$, $A_2 = \{a, e, f, g, h\}$

$A_3 = \{a, c, e, g\}$ $A_4 = \{b, d\}$ $A_5 = \{f, h\}$

Then $\{A_1, A_2\}$ is not a partition since $A_1 \cap A_2 \neq \emptyset$

Also $\{A_1, A_5\}$ is not a partition.

But,

$P = \{A_3, A_4, A_5\}$ is a partition of A since

$A_3 \cup A_4 \cup A_5 = A$ and $A_3 \cap A_4 = \emptyset$, $A_4 \cap A_5 = \emptyset$, $A_3 \cap A_5 = \emptyset$.

PARTIAL ORDERING: A relation R on a set S is called a partial ordering or partial order, if it is reflexive, anti-symmetric and transitive. A set S together with a partial ordering R is called partially ordered set, or poset, and is denoted by (S, R). Members of S are called elements of the poset.

Example: Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

Solution: Because $a \geq a$ for every integer, a, $\geq$ is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric. Finally, $\geq$ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$. It follows that $\geq$ is a partial ordering on the set of integers and $(Z, \geq)$ is a poset.

Example: $(Z^{+}, |)$ is a poset i.e divisibility relation | on set of positive integers is poset, ~~as the~~ as it is transitive, reflexive and anti-symmetric.

Example: Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution: Because $A \subseteq A$ whenever A is a subset of S, $\therefore \subseteq$ is reflexive.

It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$.

$\subseteq$ is transitive, because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence, $\subseteq$ is partial ordering on $P(S)$ and $(P(S), \subseteq)$ is a poset.

- The elements a and b of a poset $(S, \preceq)$ are called comparable if either $a \preceq b$ or $b \preceq a$. When a and b are ~~comparable~~ elements of S such that neither $a \preceq b$ nor $b \preceq a$, a and b are called incomparable.

- Example: In the poset $(Z^+, |)$ are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution: The integers 3 and 9 are comparable, because $3 | 9$. The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$.

- If $(S, \preceq)$ is poset and every two elements of S are comparable, S is called totally ordered or linearly ordered set, and $\preceq$ is called a total order or a linear order. A totally ordered set is also called a chain.

- Example: The poset $(Z, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$, whenever 'a' and 'b' are integers.

Example: The poset $(Z^+, |)$ is not totally ordered because it contains elements that are incomparable.

Lexicographic Order: Suppose we have two posets, $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$. The lexicographic ordering $\preceq$ on $A_1 \times A_2$, is defined by specifying that one pair is less than a second pair if the first entry of the first pair is less than the first entry of the second pair, or if the first entries are equal, but the second entry of this pair is less than the second entry of the second pair.

In other words, (a_1, a_2) is less than (b_1, b_2), that is

$$(a_1, a_2) \prec (b_1, b_2)$$

either if $a_1 \prec_1 b_1$ or if both $a_1 = b_1$ and $a_2 \prec_2 b_2$.

Maximal and Minimal elements:

Let $(S, \preceq_0)$ be a poset. An element 'a' is the greatest element of S if $x \leq a$ for all $x \in S$. The maximal element that exist is unique. For if a and a' are two greatest elements of S, then we ~~show~~ should have $a' \leq a$ and $a \leq a'$. Hence, $a = a'$.

Similarly, an element $b \in S$ is called least element if $b \leq x$ for all $x \in P$. The minimal element that exist is unique.

Lattices: A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a lattice. Lattices have many special properties. Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra.

Example: Determine whether $(P(S), \subseteq)$ is a lattice where S is a set.

Solution:

Let A and B be two subsets of S. The least upper bound and the greatest lower bound of A and B are $A \cup B$ and $A \cap B$, respectively, hence $(P(S), \subseteq)$ is a lattice.