Unit 2

Interpolation and Approximation

Let $f(x)$ be continuous function may be used to represent the $n+1$ data values with passing $n+1$ points. The process of computing the value of $f(x)$ or y for given x inside the given range is called Interpolation.

x	x_0	x_1	....	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	....	$f(x_n)$

Then the process of finding the value of $f(x)$ corresponding to any value of x is called interpolation.

1. ***Lagrange Interpolation***

Suppose $f(x)$ be a function with $f(x_0), f(x_1), f(x_2), \dots\dots\dots, f(x_n)$ corresponding to the values $x_0, x_1, x_2, \dots\dots\dots, x_n$ then the Lagrange's interpolation formula is given by;

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} \times f(x_0) + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_1)(x_1-x_2)\dots(x_1-x_n)} \times f(x_1) + \dots\dots$$
$$\dots\dots\dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} \times f(x_n)$$

Q. *Derive the equation for Lagrange's interpolating polynomial.*

Sol^n: Let us consider a second order polynomial of the form

$P_2(x) = b_1(x-x_0)(x-x_1) + b_2(x-x_1)(x-x_2) + b_3(x-x_2)(x-x_0)$.......... (1)

If(x_0, f_0), (x_1, f_1), (x_2, f_2) are the three interpolating points, then we have

$$P_2(x_0) = f_0 = b_2(x_0-x_1)(x_0-x_2)$$
$$P_2(x_1) = f_1 = b_3(x_1-x_2)(x_1-x_0)$$
$$P_2(x_2) = f_2 = b_1(x_2-x_0)(x_2-x_1)$$

Substituting for b_1, b_2 & b_3 in equation (1) we get,

$P_2(x) = f_0\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1\frac{(x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)} + f_2\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$ .... (2)

Equation (2) may be represented as

$$P_2(x) = f_0l_0(x) + f_1l_1(x) + f_2l_2(x)$$

$= \sum_{i=0}^{2} f_il_i(x)$ Where, $l_i(x) = \prod_{j=0,j\neq i}^{2}\frac{(x-x_j)}{(x_i-x_j)}$

In general, for n+1 points we have nth degree polynomial as $P_n(x) = \sum_{i=0}^{n} f_il_i(x)$ .... (3)

Where, $l_i(x) = \prod_{j=0,j\neq i}^{n}\frac{(x-x_j)}{(x_i-x_j)}$

The equation (3) is called the Lagrange interpolation polynomial.

Algorithm For Lagrange's interpolation

1. *Read the number of data 'n'.*
2. *Read the value at which value is needed (say y).*
3. *Read available data points x and f(x).*
4. *for i=0 to n*
   for j=0 to n
   if(i!=j)
   L[i]=L[i](y-x[j])/(x[i]-x[j])*
   end if
   end for
   end for
5. *for i=0 to n*
   *sum=sum+L[i]*f[i]*
   end for
6. *Print interpolation value 'sum' at y.*
7. *Stop*

Examples

***1.* Use Lagrange's interpolation formula find the value of f(x) at x= 10 i.e. f(10) from following data.**

x	5	6	9	11
f(x)	12	13	14	16

Sol^n:

Here, $x_0 = 5, x_1 = 6, x_2 = 9, x_3 = 11$

$f(x_0) = 12, f(x_1) = 13, f(x_2) = 14, f(x_3) = 16$

By Lagrange's interpolation; we have,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3)$$

$$\therefore f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13 + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

= 2 - 4.3333 + 11.6666 + 5.3333

=14.6666

***2.* Find the Lagrange interpolation polynomial to fit the following data.**

x_i	0	1	2	3
e^{x_i}	0	1.7183	6.3891	19.0855

Estimate the value of $e^{1.9}$.

Sol^n:

Here, $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$

$f(x_0) = 0, f(x_1) = 1.7183, f(x_2) = 6.3891, f(x_3) = 19.0855$

By Lagrange's interpolation; we have,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3)$$

$$or, f(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} \times 0 + \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} \times 1.7183 + \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} \times 6.3891 + \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} \times 19.0855$$

Now, to estimate the value of $e^{1.9}$,

$$\therefore f(1.9) = \frac{(1.9-1)(1.9-2)(1.9-3)}{(0-1)(0-2)(0-3)} \times 0 + \frac{(1.9-0)(1.9-2)(1.9-3)}{(1-0)(1-2)(1-3)} \times 1.7183 + \frac{(1.9-0)(1.9-1)(1.9-3)}{(2-0)(2-1)(2-3)} \times 6.3891 + \frac{(1.9-0)(1.9-1)(1.9-2)}{(3-0)(3-1)(3-2)} \times 19.0855$$

= 0 + 0.1796 + 6.0089 – 0.5439

= 5.6446

$\therefore$ $\boldsymbol{e^{1.9}}$ = 5.6446

3. ***Find the value of $f(x)$ at $x = 1$ for the following data:***

x	-1	-2	2	4
$f(x)$	-1	-9	11	69

Sol^n:

Here, $x_0 = -1, x_1 = -2, x_2 = 2, x_3 = 4$

$f(x_0) = -1, f(x_1) = -9, f(x_2) = 11, f(x_3) = 69$

By Lagrange's interpolation; we have,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3)$$

$$\therefore f(1) = \frac{(1+2)(1-2)(1-4)}{(-1+2)(-1-2)(-1-4)}(-1) + \frac{(1+1)(1-2)(1-4)}{(-2+1)(-2-2)(-2-4)}(-9) + \frac{(1+1)(1+2)(1-4)}{(2+1)(2+2)(2-4)}(11) + \frac{(1+1)(1+2)(1-2)}{(4+1)(4+2)(4-2)}(69)$$

= - 0.6 + 2.25 + 8.25 - 6.9

= 3

$\therefore f(1) = 3$

2. Newton's Divide Difference Interpolation Method

Suppose $f(x)$ be a function with $f(x_0), f(x_1), f(x_2), \dots\dots\dots, f(x_n)$ corresponding to the values $x_0, x_1, x_2, \dots\dots\dots, x_n$ then Newton divided difference interpolation is given by;

$$f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\ + \cdots \dots\dots\dots (x - x_0)(x - x_1) \dots\dots (x - x_n)f[x_0, x_1, \dots\dots x_n]$$

Or, it can be also written as;

$$f(x) = f_0 + (x - x_0)\Delta f_0 + (x - x_0)(x - x_1)\Delta^2 f_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 f_0 + \cdots \dots$$

Now, we can construct the divided difference table as follows;

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	f_0			
		$\frac{f_1 - f_0}{x_1 - x_0} = \Delta f_0$		
x_1	f_1		$\frac{\Delta f_1 - \Delta f_0}{x_2 - x_0} = \Delta^2 f_0$	
		$\frac{f_2 - f_1}{x_2 - x_1} = \Delta f_1$		$\frac{\Delta^2 f_1 - \Delta^2 f_0}{x_3 - x_0} = \Delta^3 f_0$
x_2	f_2		$\frac{\Delta f_2 - \Delta f_1}{x_3 - x_1} = \Delta^2 f_1$	
		$\frac{f_3 - f_2}{x_3 - x_2} = \Delta f_2$		
x_3	f_3			

Algorithm for Newton Interpolation

1. *Read the number of data 'n'.*
2. *Read the value at which value is needed (say y).*
3. *Read the values of x_i and f_i.*
4. *Initialize sum=f_0, mult=1*
5. *for i=0 to n*
   {
   for j=0 to n
   {
   $f_j = (f_{j+1} - f_j)/(x_{j+1} - x_j)$
   }
   Repeat j
   Mult=(y-x_i)*
   *Sum+=f_i*mult*
   }
   Repeat i
6. *Print functional value 'sum' at y.*
7. *Stop*

Examples

***1.* Using the divide difference table (Newton's divided difference interpolation). Find the value of $f(1.75)$.**

x	1.1	2.0	3.5	5	7.1
f	0.6981	1.4715	2.1287	2.0521	1.4480

Sol^n^:

The divide difference table for given data;

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
1.1	0.6981				
		$\frac{1.4715-0.6981}{2.0-1.1} = 0.8593$			
2.0	1.4715		$\frac{0.4381-0.8593}{3.5-1.1} = -0.1755$		
		$\frac{2.1287-1.4715}{3.5-2.0} = 0.4381$		$\frac{-0.1630+0.1755}{5-1.1} = 0.0031$	
3.5	2.1287		$\frac{-0.0510-0.4381}{5-2.0} = -0.1630$		$\frac{0.0190-0.0031}{7.1-1.1} = 0.0026$
		$\frac{2.0521-2.1287}{5-3.5} = -0.0510$		$\frac{-0.0657+0.1630}{7.1-2.0} = 0.0190$	
5	2.0521		$\frac{-0.2876+0.0510}{7.1-3.5} = -0.0657$		
		$\frac{1.4480-2.0521}{7.1-5} = -0.2876$			
7.1	1.4480				

We have,

$$f(x) = f(x_0) + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3] + (x-x_0)(x-x_1)(x-x_2)(x-x_3)f[x_0,x_1,x_2,x_3,x_4]$$

$$f(1.175) = 0.6981 + (1.175-1.1)0.8593 + (1.175-1.1)(1.175-2.0)(-0.1755) + (1.175-1.1)(1.175-2.0)(1.175-3.5)0.0031 + (1.175-1.1)(1.175-2.0)(1.175-3.5)(1.175-5)0.0026$$

$= 0.7719$

2. ***Using Newton's divide difference interpolating polynomial estimate the value of $f(x)$ at $x = 4$ for the function defined as***

x	***0***	***2***	***3***	***6***
f	***648***	***704***	***729***	***792***

Sol:

The divide difference table for given data;

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$
0	648			
		$\frac{704-648}{2-0} = 28$		
2	704		$\frac{25-28}{3-0} = -1$	
		$\frac{729-704}{3-2} = 25$		$\frac{-1+1}{6-0} = 0$
3	729		$\frac{21-25}{6-2} = -1$	
		$\frac{792-729}{6-3} = 21$		
6	792			

We have,

$$f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3]$$

$$= 648 + (x - 0)28 + (x - 0)(x - 2)(-1) + 0$$

$$\therefore f(4) = 648 + (4 - 0)28 + (4 - 0)(4 - 2)(-1)$$

$$= 752$$

Methods to find the interpolation with equal intervals

1. Newton Forward Interpolation method

2. Newton Backward Interpolation method

Newton Forward Interpolation method

We know that, Newton's Forward Interpolation formula as

$$y_s = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 y_0 + \cdots \ldots$$

Where, $s = \frac{x_s - x_0}{h}$

$y_0 = f(x_0), y_1 = f(x_1), \ldots\ldots, y_n = f(x_n)$

x_s= Value at which interpolation is to be found

x_0= Initial value

h= Interval of 'x'

The difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		$\Delta y_0 = y_1 - y_0$		
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		
x_3	y_3			

Newton Backward Interpolation method

If the table is too long and if the required point is close to the end point of the table, we can use newton backward interpolation formula.

We know that, Newton's backward Interpolation formula as

$$y_s = y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 y_n + \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 y_n + \cdots \ldots.$$

Where, $s = \frac{x_s - x_n}{h}$

x_s= Value at which interpolation is to be found

x_n= Final value

h= Interval of 'x'

The difference table is:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		$\nabla y_0 = y_1 - y_0$		
x_1	y_1		$\nabla^2 y_0 = \nabla y_1 - \nabla y_0$	
		$\nabla y_1 = y_2 - y_1$		$\nabla^3 y_0 = \nabla^2 y_1 - \nabla^2 y_0$
x_2	y_2		$\nabla^2 y_1 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_3 - y_2$		
x_3	y_3			

Examples

***1.* Find the functional value at x = 25 from the following data using forward difference table.**

x	***10***	***20***	***30***	***40***	***50***
y	***0.1736***	***0.3420***	***0.5000***	***0.6428***	***0.7660***

Soln:

The difference table is;

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	0.1736				
		0.1684			
20	0.3420		-0.0104		
		0.1580		0.0048	
30	0.5000		-0.0152		-0.0004
		0.1428		0.0044	
40	0.6428		-0.0196		
		0.1232			
50	0.7660				

Here,
x_0=10
h=10
$s = \frac{x_s - x_0}{h} = \frac{25-10}{10} = 1.5$
Now according to Newton's forward difference formula;

$$y_s = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 y_0$$

$$y(25) = 0.1736 + (1.5)(0.1684) + \frac{1.5(1.5-1)}{2!}(-0.0104) + \frac{1.5(1.5-1)(1.5-2)}{3!}(0.0048) + \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{4!}(-0.0004)$$

=0.4220

$\therefore y(25) = 0.4220$

***2.* Construct Newton forward interpolation formula from given table to evaluate $f(5)$.**

x	***4***	***6***	***8***	***10***
y	***1***	***3***	***8***	***16***

Soln:

The difference table is;

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

Here,
x_0=4
h=2
$s = \frac{x_s - x_0}{h} = \frac{5-4}{2} = 0.5$
Now according to Newton's forward difference formula;
$y_s = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0$
$y_5 = 1 + (0.5)(2) + \frac{0.5(0.5-1)}{2!}(3) + \frac{0.5(0.5-1)(0.5-2)}{3!}(0)$
= 1+ 1 – 0.375 + 0
= 1.625

$\therefore f(5) = 1.625$

3. ***Find the functional value at x = 3.6 from the following data using forward difference table.***

x	***2***	***2.5***	***3***	***3.5***	***4***	***4.5***
f(x)	***1.43***	***1.03***	***0.76***	***0.6***	***0.48***	***0.39***

Sol^n:

x	*y= f(x)*	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2	1.43					
		-0.4				
2.5	1.03		0.13			
		-0.27		-0.02		
3	0.76		0.11		-0.05	
		-0.16		-0.07		0.11
3.5	0.6		0.04		0.06	
		-0.12		-0.01		
4	0.48		0.03			
		-0.09				
4.5	0.39					

Here,
x_0=2
h=0.5
$s = \frac{x_s - x_0}{h} = \frac{3.6-2}{0.5} = 3.2$

Now according to Newton's forward difference formula;

$$y_s = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 y_0 + \frac{s(s-1)(s-2)(s-3)(s-4)}{5!}\Delta^5 y_0$$

$$y_{3.6} = 1.43 + 3.2(-0.4) + \frac{3.2(3.2-1)}{2!}(0.13) + \frac{3.2(3.2-1)(3.2-2)}{3!}(-0.02) + \frac{3.2(3.2-1)(3.2-2)(3.2-3)}{4!}(-0.05) + \frac{3.2(3.2-1)(3.2-2)(3.2-3)(3.2-4)}{5!}(0.11)$$

= 1.43 – 1.28 + 0.4576 – 0.02816 – 0.00352 – 0.001239

= 0.574681

4. Find the value of f(x) at x=17 using Newton's backward interpolation method for the following data.

x	*0*	*5*	*10*	*15*	*20*
f(x)	***1.0***	***1.6***	***3.8***	***8.2***	***15.4***

Sol^n:

x	*y=f(x)*	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	1.0				
		0.6			
5	1.6		1.6		
		2.2		0.6	
10	3.8		2.2		0
		4.4		0.6	
15	8.2		2.8		
		7.2			
20	15.4				

Here,

$x_n = x_4 = 20$

$h = 5$

$s = \frac{x_s - x_n}{h} = \frac{17-20}{5} = -0.6$

Now according to Newton's backward difference formula;

$$y_s = y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 y_n + \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 y_n$$

$$\therefore y_{17} = 15.4 + (-0.6)(7.2) + \frac{(-0.6)(-0.6+1)}{2!}(2.8) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(0.6) + 0$$

$= 15.4 - 4.32 - 0.336 - 0.0336 + 0$

$= 10.7104$

$\therefore y_{17} = 10.7104$

5. Estimate the value of ln(3.5) using Newton's backward difference formula, given the following data

x	*1.0*	*2.0*	*3.0*	*4.0*
ln(x)	***0.0***	***0.6931***	***1.0986***	***1.3863***

Solution:

x	$y=ln(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$
1.0	0.0			
		0.6931		
2.0	0.6931		-0.2876	
		0.4055		0.1698
3.0	1.0986		-0.1178	
		0.2877		
4.0	1.3863			

Here,

$x_n = x_3 = 4.0$
$h=1.0$
$s = \frac{x_s - x_n}{h} = \frac{3.5-4.0}{1.0} = -0.5$
Now according to Newton's backward difference formula;
$y_s = y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 y_n$
$y_{3.5} = 1.3863 + (-0.5)(0.2877) + \frac{(-0.5)(-0.5+1)}{2!}(-0.1178) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(0.1698)$
$= 1.3863 - 0.14385 + 0.014725 - 0.021225$
$= 1.23595$

$\therefore \ln(3.5) = 1.23595$

Cubic Spline Interpolation

Cubic interpolation works by constructing the (cubic) polynomial in pieces. Given n points will construct n-1 different (cubic) polynomials. These polynomial have consistent derivatives at the end points.

Formula
Formula 1:
$h_i a_{i-1} + 2a_i(h_i + h_{i+1}) + h_{i+1}a_{i+1} = 6\left[\frac{f_{i+1}-f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right]$.................. (1)
Where $a_0 = a_n = 0$

Formula 2:
$s_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 U_i - U_i^3) + \frac{a_i}{6h_i}(U_{i-1}^3 - h_i^2 U_{i-1}) + \frac{1}{h_i}(f_i U_{i-1} - f_{i-1} U_i)$.......... (2)

$h_i = x_i - x_{i-1}$ and $U_i = x - x_i$

- Number of coefficient is equal to number of points.
- Evaluate equation (1) for $i = 1 \ to \ n-1$
- Now evaluate equation (2) at $i = a$ value given by position of interval.

Examples

1. *Given the data points*

x	4	9	16
f	2	3	4

Estimate the functional value f at x=7 using cubic splines.

Soln:

Here,

$h_1 = x_1 - x_0$=9-4=5

$h_2 = x_2 - x_1$=16-9=7

f_0=2, f_1=3, f_2=4

$a_0 = a_2$ =0

We have,

$$h_i a_{i-1} + 2a_i(h_i + h_{i+1}) + h_{i+1}a_{i+1} = 6\left[\frac{f_{i+1}-f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right]$$

For $i = 1$

$$h_1 a_0 + 2a_1(h_1 + h_2) + h_2 a_2 = 6\left[\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right]$$

$$0 + 2a_1(5+7) + 0 = 6\left[\frac{4-3}{7} - \frac{3-2}{5}\right]$$

$$24a_1 = 6\left[\frac{1}{7} - \frac{1}{5}\right]$$

$\boldsymbol{a_1}$**= -0.0143**

Again from second formula,

$$s_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 U_i - U_i^3) + \frac{a_i}{6h_i}(U_{i-1}^3 - h_i^2 U_{i-1}) + \frac{1}{h_i}(f_i U_{i-1} - f_{i-1}U_i)$$

$$s_1(x) = \frac{a_0}{6h_1}(h_1^2 U_1 - U_1^3) + \frac{a_1}{6h_1}(U_0^3 - h_1^2 U_0) + \frac{1}{h_1}(f_1 U_0 - f_0 U_1)$$

$U_0 = x - x_0 = x - 4$

$U_1 = x - x_1 = x - 9$

$$\therefore s_1(7) = 0 + \frac{-0.0143}{6X5}[(7-4)^3 - 5^2(7-4)] + \frac{1}{5}[3(7-4) - 2(7-9)]$$

=2.6229

2. *Estimate $f(3)$ from the following data using cubic spline interpolation.*

x	1	2.5	4	5.7
$f(x)$	-2.0	4.2	14.4	31.2

Solⁿ:

$h_1 = x_1 - x_0 = 2.5 - 1 = 1.5$
$h_2 = x_2 - x_1 = 4 - 2.5 = 1.5$
$h_3 = x_3 - x_2 = 5.7 - 4 = 1.7$
$f_0 = -2.0,\ f_1 = 4.2,\ f_2 = 14.4,\ f_3 = 31.2$
$a_0 = a_3 = 0$
We have,

$$h_i a_{i-1} + 2a_i(h_i + h_{i+1}) + h_{i+1}a_{i+1} = 6\left[\frac{f_{i+1}-f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right]$$

For $i = 1$,

$$h_1 a_0 + 2a_1(h_1 + h_2) + h_2 a_2 = 6\left[\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right]$$

$$0 + 2a_1(1.5+1.5) + 1.5a_2 = 6\left[\frac{14.4-4.2}{1.5} - \frac{4.2+2.0}{1.5}\right]$$

$$\mathbf{6a_1 + 1.5a_2 = 16} \ \ldots\ldots\ldots\ldots\ldots \ \mathbf{(1)}$$

For $i = 2$,

$$h_2 a_1 + 2a_2(h_2 + h_3) + h_3 a_3 = 6\left[\frac{f_3 - f_2}{h_3} - \frac{f_2 - f_1}{h_2}\right]$$

$$1.5a_1 + 2a_2(1.5+1.7) + 0 = 6\left[\frac{31.2-14.4}{1.7} - \frac{14.4-4.2}{1.5}\right]$$

$$\mathbf{1.5a_1 + 6.4a_2 = 18.494} \ \ldots\ldots\ldots\ldots\ldots \ \mathbf{(2)}$$

Solving equation (1) & (2) we get,
$a_1 = 2.065$
$a_2 = 2.406$

Now,

$$s_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 U_i - U_i^3) + \frac{a_i}{6h_i}(U_{i-1}^3 - h_i^2 U_{i-1}) + \frac{1}{h_i}(f_i U_{i-1} - f_{i-1}U_i)$$

$$s_2(x) = \frac{a_1}{6h_2}(h_2^2 U_2 - U_2^3) + \frac{a_2}{6h_2}(U_1^3 - h_2^2 U_1) + \frac{1}{h_2}(f_2 U_1 - f_1 U_2)$$

$U_1 = x - x_1 = x - 2.5$

$U_2 = x - x_2 = x - 4$

$$s_2(x) = \frac{2.065}{6\text{x}1.5}(1.5^2(x-4) - (x-4)^3) + \frac{2.406}{6\text{x}1.5}\big((x-2.5)^3 - 1.5^2(x-2.5)\big) + \frac{1}{1.5}(14.4(x-2.5) - 4.2(x-4))$$

$$\therefore s_2(3) = \frac{2.065}{9}(2.25(3-4) - (3-4)^3) + \frac{2.406}{9}\big((3-2.5)^3 - 2.25(3-2.5)\big) + \frac{1}{1.5}(14.4(3-2.5) - 4.2(3-4))$$

$$= 7.0793$$

$\therefore \boldsymbol{f(3) = 7.0793}$

Curve Fitting

Curve fitting is the process of introducing mathematical relationship between dependent and independent variables in the form of an equation for a given set of data.

Fitting of a straight line

$y = a + bx$ ……… (i) where a and b are constant and are unknown.

The normal equation of (i) is

$\sum y_i = na + b\sum x_i$ …….. (ii)

$\sum x_i y_i = a\sum x_i + b\sum x_i^2$ …….. (iii)

Solving for a and b, we get

$$b = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a = \frac{\sum y_i}{n} - b\frac{\sum x_i}{n} = \bar{y} - b\bar{x}$$

Algorithm for linear Regression

1. *Read values of n, x_i, y_i*
2. *Compute sum of powers and products*
   $\sum x_i, \sum y_i, \sum x_i^2, \sum x_i y_{i_i}$
3. *Compute:*
   $b = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2}$ *and*
   $a = \bar{y} - b\bar{x}$
4. *Check whether the denominator of the equation for b is zero. If not compute a and b.*
5. *Print out the equation.*
6. *Stop*

Examples

1. *Fit a straight line to the following set of data.*

x	***1***	***2***	***3***	***4***	***5***
y	***3***	***4***	***5***	***6***	***8***

Sol^n:

x	y	x^2	xy
1	3	1	3
2	4	4	8
3	5	9	15
4	6	16	24
5	8	25	40
$\sum x = 15$	$\sum y = 26$	$\sum x^2 = 55$	$\sum xy = 90$

We have,

$$b = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} = \frac{5\times 90 - 15\times 26}{5\times 55 - 15^2} = 1.2$$

$$a = \frac{\sum y_i}{n} - b\frac{\sum x_i}{n} = \frac{26}{5} - 1.2\times\frac{15}{5} = 1.6$$

Therefore the linear equation is

$$y = 1.6 + 1.2x$$

2. *Given the data set (x_i, y_i) as (20.5, 765), (32.7, 826),(51.0, 873), (73.2, 942), (95.7, 1032) find the linear least square to fit given data.*

Soln:

x	y	x^2	xy
20.5	765	420.25	15682.5
32.7	826	1069.29	27010.2
51.0	873	2601	44523
73.2	942	5358.24	68954.4
95.7	1032	9158.49	98762.4
$\sum x = 273.1$	$\sum y = 4438$	$\sum x^2 = 18607.27$	$\sum xy = 254932.5$

We have,

$$b = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} = \frac{5\times 254932.5 - 273.1\times 4438}{5\times 18607.27 - (273.1)^2} = 3.3949$$

$$a = \frac{\sum y_i}{n} - b\frac{\sum x_i}{n} = \frac{4438}{5} - 3.3949\times\frac{273.1}{5} = 702.171$$

Therefore the linear equation is

$$y = 702.171 + 3.3949x$$

3. *Fit the following set of data to a curve of the form $y = ab^x$.*

x	*2*	*4*	*6*	*8*	*10*	*12*
y	*16*	*17.1*	*8.7*	*6.4*	*4.7*	*2.6*

Sol^n:

Given that,

$y = ab^x$ …….. (i)

Taking log on both side; we have,

$\log y = \log a + x\log b$ ………. (ii)

Comparing equation (ii) with $Y = A + Bx$

$Y = \log y$

$B = \log b$

$A = \log a$

X	y	Y=logy	X^2	XY
2	16	1.20412	4	2.40824
4	11.1	1.04532	16	4.18128
6	8.7	0.93952	36	5.63712
8	6.4	0.80618	64	6.44944
10	4.7	0.67209	100	6.7209
12	2.6	0.41497	144	4.97964
$\sum X = 42$		$\sum Y = 5.0822$	$\sum X^2 = 364$	$\sum XY = 30.37662$

We have,

$$B = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} = \frac{6 \times 30.37662 - 42 \times 5.0822}{6 \times 364 - 42^2} = -0.07427$$

$$A = \frac{\sum y_i}{n} - B\frac{\sum x_i}{n} = \frac{5.0822}{6} - (-0.07427) \times \frac{42}{6} = 1.36692$$

Now,

$a = \ antilog(1.36692) = 23.27662$

$b = antilog(-0.07427) = 0.84281$

So the equation (i) becomes

$$y = 23.27662 \times 0.84281^x$$

---

4. *Fit the following set of data to a curve of the form $y = ae^{bx}$.*

x	***2***	***4***	***6***	***8***	***10***	***12***
y	***16***	***17.1***	***8.7***	***6.4***	***4.7***	***2.6***

Sol^n:

The given curve is $y = ae^{bx}$……….. (i)

Taking log on both sides, we have,

$logy = loga + bxloge$ ……….. (ii)

Comparing equation (ii) with $Y = A + Bx$

$Y = \log y$

$B = b\log e$

$A = \log a$

X	y	Y=logy	X^2	XY
2	16	1.20412	4	2.40824
4	11.1	1.04532	16	4.18128
6	8.7	0.93952	36	5.63712
8	6.4	0.80618	64	6.44944
10	4.7	0.67209	100	6.7209
12	2.6	0.41497	144	4.97964
$\sum X = 42$		$\sum Y = 5.0822$	$\sum X^2 = 364$	$\sum XY = 30.37662$

We have,

$$B = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} = \frac{6\times 30.37662 - 42\times 5.0822}{6\times 364 - 42^2} = -0.07427$$

$$A = \frac{\sum y_i}{n} - B\frac{\sum x_i}{n} = \frac{5.0822}{6} - (-0.07427)\times\frac{42}{6} = 1.36692$$

Now,

$$a = \ antilog(1.36692) = 23.27662$$
$$b = \frac{B}{loge} = \frac{-0.07427}{loge} = -0.17101$$

The equation (i) becomes

$$y = 23.27662e^{-0.17101x}$$

Fitting quadratic polynomial

The equation for quadratic polynomial is

$y = a_1 + a_2 x + a_3 x^2 \ldots\ldots\ldots.$ (i)

The normal form equation of (i) are

$na_1 + a_2\sum x_i + a_3\sum x_i^2 = \sum y_i \ \ldots\ldots\ldots.$ (ii)

$a_1\sum x_i + a_2\sum x_i^2 + \ a_3\sum x_i^3 = \sum x_i y_i \ \ldots\ldots\ldots.$ (iii)

$a_1\sum x_i^2 + a_2\sum x_i^3 + a_3\sum x_i^4 = \sum x_i^2 y_i \ \ldots\ldots\ldots.$ (iv)

These equation can be expressed in matrix form as

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix}$$

Examples

1. Find the best fitting quadratic polynomial from following data using least square approximation.

x	***-2***	***-1.2***	***0***	***1***	***1.2***	***2.5***	***3***	***4.5***	***6.3***
y	***10.39***	***2.96***	***-2.0***	***-2.63***	***-2.46***	***0.83***	***3.1***	***12.8***	***30.4***

Sol^n:

Given no. of data $(n) = 9$

The equation for quadratic polynomial is

$y = a_1 + a_2x + a_3x^2$……… (i)

The given equation in regression matrix form is

$$\begin{bmatrix} n & \sum x & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix} \text{………. (ii)}$$

The various summations are given below,

$\sum x_i = 19.3$

$\sum y_i = 53.39$

$\sum x_i^2 = 86.07$

$\sum x_i^3 = 376.79$

$\sum x_i^4 = 708.5683$

$\sum x_i y_i = 230.5810$

$\sum x_i^2 y_i = 1538.5135$

> ***Note:*** To find the value of $\sum x_i, \sum y_i, \sum x_i^2, \sum x_i^3, \sum x_i^4, \sum x_i y_i, \sum x_i^2 y_i$ using calculator (CASIO fx-991ES) perform following steps:
> 1. Press→(mode) and 3
> 2. Select 3 and put the value of *x* and *y* and press AC.
> 3. Press :(shift) and (stat i.e. 1)
> 4. Select 3: Sum and take the required value.

Now from (ii)

$$\begin{bmatrix} 9 & 19.3 & 86.07 \\ 19.3 & 86.07 & 376.79 \\ 86.07 & 376.79 & 708.5683 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 53.39 \\ 230.5810 \\ 1538.5135 \end{bmatrix}$$

i.e.

$9a_1 + 19.3a_2 + 86.07a_3 = 53.39$

$19.3a_1 + 86.07a_2 + 376.79a_3 = 230.5810$

$86.07a_1 + 376.79a_2 + 708.5683a_3 = 1538.5135$

Solving, we get

{You can use calculator to solve these equations.

$a_1 = 0.5506$, $a_2 = 5.0132$, $a_3 = -0.5614$

The equation (i) becomes

$$y = 0.5506 + 5.0132x - 0.5614x^2$$

Which is required quadratic polynomial.

2. Fit a quadratic polynomial to the following set of data using least square approximation.

x	*1*	*2*	*3*	*4*	*5*
y	*10*	*12*	*8*	*10*	*14*

Sol^n:

Given no. of data (n) =5

The equation for quadratic polynomial is

$y = a_1 + a_2x + a_3x^2$......... (i)

The given equation in regression matrix form is

$$\begin{bmatrix} n & \sum x & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix} \text{.......... (ii)}$$

The various summations are given below,

$\sum x_i = 15$, $\sum y_i = 54$, $\sum x_i^2 = 55$,

$\sum x_i^3 = 225$, $\sum x_i^4 = 979$, $\sum x_i y_i = 168$, $\sum x_i^2 y_i = 640$

Now from (ii)

$$\begin{bmatrix} 5 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 54 \\ 168 \\ 640 \end{bmatrix}$$

i.e.

$5a_1 + 15a_2 + 55a_3 = 54$
$15a_1 + 55a_2 + 225a_3 = 168$
$55a_1 + 225a_2 + 979a_3 = 640$

Solving, we get

$a_1 = 13$, $a_2 = -3.686$, $a_3 = 0.714$

The equation (i) becomes

$$y = 13 - 3.686x + 0.714x^2$$

Which is required quadratic polynomial.

References:

- *E. Balagurusamy, Numerical Methods, Tata McGraw-Hill*

Please let me know if I missed anything or anything is incorrect.

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