Unit-3

Numerical Differentiation and Integration

Numerical Differentiation

The method of obtaining the derivative of a function using a numerical technique is known as numerical differentiation. There are essentially two situations where numerical differentiation is required. They are:

1. The function values are known but the function is unknown. Such functions are called tabulated functions.
2. The function to be differentiated is complicated and, therefore, it is difficult to differentiate.

Differentiating continuous function

$f'(x) =?$ *of a function* $f(x)$, *when the function itself is available.*

- ***Forward Difference Quotient***

We have, Taylor series is;

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots\ldots \quad \text{(i)}$$

Neglecting higher order of derivative, we get,

$$f(x+h) = f(x) + hf'(x)$$

$$\boxed{\therefore f'(x) = \frac{f(x+h)-f(x)}{h}}$$

Which is the first order *forward difference quotient*.

- ***Backward Difference Quotient***

We have, Taylor series is;

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots\ldots \quad \text{(i)}$$

Neglecting higher order of derivative, we get,

$$f(x-h) = f(x) - hf'(x)$$

$$\boxed{\therefore f'(x) = \frac{f(x)-f(x-h)}{h}}$$

Which is the first order *backward difference quotient*.

- ***Central Difference Quotient***

We have, Taylor series is;

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots\ldots$ (i)

Similarly,

$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots\ldots$ (ii)

Subtracting eq. (ii) from eq. (i) and neglecting higher order derivative we get.

$$\boxed{f'(x) = \frac{f(x+h)-f(x-h)}{2h}}$$

Which is the *central difference quotient*.

Examples

1. Estimate approximate derivative of $f(x) = x^2$ at $x = 1$, for h=0.2 using the forward difference and central difference formula.

Sol^n^:

Using the forward difference formula:

We have,

$f'(x) = \frac{f(x+h)-f(x)}{h}$

Therefore,

$f'(1) = \frac{f(1+0.2)-f(1)}{0.2} = \frac{f(1.2)-f(1)}{0.2} = 2.2$

Using the central difference formula:

We have,

$f'(x) = \frac{f(x+h)-f(x-h)}{2h}$

Therefore,

$f'(1) = \frac{f(1+0.2)-f(1-0.2)}{2\times0.2} = \frac{f(1.2)-f(0.8)}{0.4} = 2$

2. Estimate the first derivative of f(x)=ln x at x=1 using the second order central difference formula.

<u>Sol</u>

Given,

f(x)=ln x

Take h=0.1

We have,

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h}$$

$$\therefore f'(1) = \frac{f(1+0.1)-f(1-0.1)}{2*0.1} = \frac{f(1.1)-f(0.9)}{0.2} = = \frac{0.095-(-0.1054)}{0.2} = 1.002$$

Higher- order Derivatives

We have, Taylor series is;

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots\dots\dots \qquad \text{(i)}$$

Similarly,

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \dots\dots\dots \qquad \text{(ii)}$$

Adding eq. (i) and eq. (ii) by neglecting higher derivatives, we get,

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x)$$

$$\boxed{\therefore f''(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2}}$$

Example

Q. Find approximation to second derivative of cos(x) at x=0.75 with h=0.01.

Sol:

We have,

$$f''(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$$

$$\therefore f''(0.75) = \frac{f(0.76)-2f(0.75)+f(0.74)}{0.0001} = -0.000305$$

Derivative Using Newton Forward Interpolation Formula

Newton forward interpolation formula is,

$$y(x) = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \cdots \ldots \ldots \ldots ..$$

Where, $s = \frac{x-x_0}{h} \Longrightarrow x = x_0 + sh$

$$y(x_0 + sh) = y_0 + s\Delta y_0 + \frac{s^2-s}{2!}\Delta^2 y_0 + \frac{s^3-3s^2+2s}{3!}\Delta^3 y_0 + \cdots \ldots \ldots \ldots ..$$

Differentiating w.r.to. 's' we get,

$$y'(x) = \frac{dy}{dx}h = \Delta y_0 + \frac{(2s-1)}{2!}\Delta^2 y_0 + \frac{3s^2-6s+2}{3!}\Delta^3 y_0 + \cdots \ldots \ldots ....$$

$$\boxed{\therefore \frac{dy}{dx} = y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{(2s-1)}{2!}\Delta^2 y_0 + \frac{3s^2-6s+2}{3!}\Delta^3 y_0 + \cdots \ldots \ldots\right]}$$

Differentiating Again, we get,

$$\boxed{\frac{d^2y}{dx^2} = y''(x) = \frac{1}{h^2}[\Delta^2 y_0 + (s-1)\Delta^3 y_0 + \cdots \ldots \ldots]}$$

Example

Q. ***The distance travelled by a car at various time intervals are given as follows:***

t(sec)	*1.5*	*2.0*	*2.5*	*3.0*	*3.5*	*4.0*
y(meter)	*3.375*	*7.0*	*13.625*	*24*	*38.875*	*59*

Evaluate the velocity and acceleration of the car at t=1.5 sec.

Soln:

The forward difference table for given data

t	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.5	3.375					
		3.625				
2.0	7.0		3.0			
		6.625		0.75		
2.5	13.625		3.75		0	
		10.375		0.75		0
3.0	24		4.5		0	
		14.875		0.75		
3.5	38.875		5.25			
		20.125				
4.0	59					

Here,

$x_0 = 1.5$, x =1.5, h = 0.5, $s = \frac{x-x_0}{h} = \frac{1.5-1.5}{0.5} = 0$

We have,

$\frac{dy}{dx} = y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{(2s-1)}{2!}\Delta^2 y_0 + \frac{3s^2-6s+2}{3!}\Delta^3 y_0 + \cdots \ldots \ldots\right]$

$\therefore\ (\frac{dy}{dx})_{1.5} = y'(1.5) = \frac{1}{0.5}\left[3.625 + \left(-\frac{1}{2}\right) \times 3 + \frac{2}{6}(0.75)\right]$

= 4.75 m/sec

Again,

$\frac{d^2y}{dx^2} = y''(x) = \frac{1}{h^2}[\Delta^2 y_0 + (s-1)\Delta^3 y_0 + \cdots \ldots \ldots]$

$\therefore\ (\frac{d^2y}{dx^2})_{1.5} = y''(1.5) = \frac{1}{0.5^2}[3.0 + (-1) \times 0.75]$

=9 m/sce^2

Numerical Integration

The process of evaluating a definite integral from a set of tabulated value of the integral $f(x)$ is called numerical integration.

$I = \int_{x_0}^{x_n} f(x)dx$

Newton's cotes formula

Let $y = f(x)$ be a function and $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \ldots\ldots\ldots\ldots\ldots, y_n = f(x_n)$.

Where, $x_n = x_0 + nh$

By Newton forward interpolation formula,

$f(x) = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \cdots \ldots \ldots \ldots ..$

Where, $s = \frac{x-x_0}{h} \Rightarrow x = x_0 + sh$

Now,

$\int_{x_0}^{x_n} ydx = \int_{x_0}^{x_0+nh} f(x)dx$

$$\boxed{\begin{array}{c}\text{Put } x = x_0 + sh \Rightarrow dx = h\,ds \\ \boxed{\Rightarrow s = 0 \text{ to } s = n}\end{array}}$$

$= \int_0^n f(x_0 + sh)h\,ds$

$$\int_{x_0}^{x_n} ydx = h\int_0^n \left[y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \cdots \ldots\right] ds$$

$$= h\int_0^n \left[y_0 + s\Delta y_0 + \left(\frac{s^2-s}{2}\right)\Delta^2 y_0 + \left(\frac{s^3-3s^2+2s}{6}\right)\Delta^3 y_0 + \cdots \ldots\right] ds$$

$$= h\left[sy_0 + \frac{s^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{s^3}{3} - \frac{s^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{s^4}{4} - \frac{3s^3}{3} + \frac{2s^2}{2}\right)\Delta^3 y_0 + \cdots \ldots\right]_0^n$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{12}(2n^3 - 3n^2)\Delta^2 y_0 + \frac{1}{24}(n^4 - 4n^3 + 4n^2)\Delta^3 y_0 + \cdots \ldots\right]$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{1}{12}(2n^2 - 3n)\Delta^2 y_0 + \frac{1}{24}(n^3 - 4n^2 + 4n)\Delta^3 y_0 + \cdots \ldots\right]$$

$$\boxed{\therefore \int_{x_0}^{x_n} ydx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{1}{12}(2n^2 - 3n)\Delta^2 y_0 + \frac{1}{24}(n^3 - 4n^2 + 4n)\Delta^3 y_0 + \cdots \ldots\right]}$$

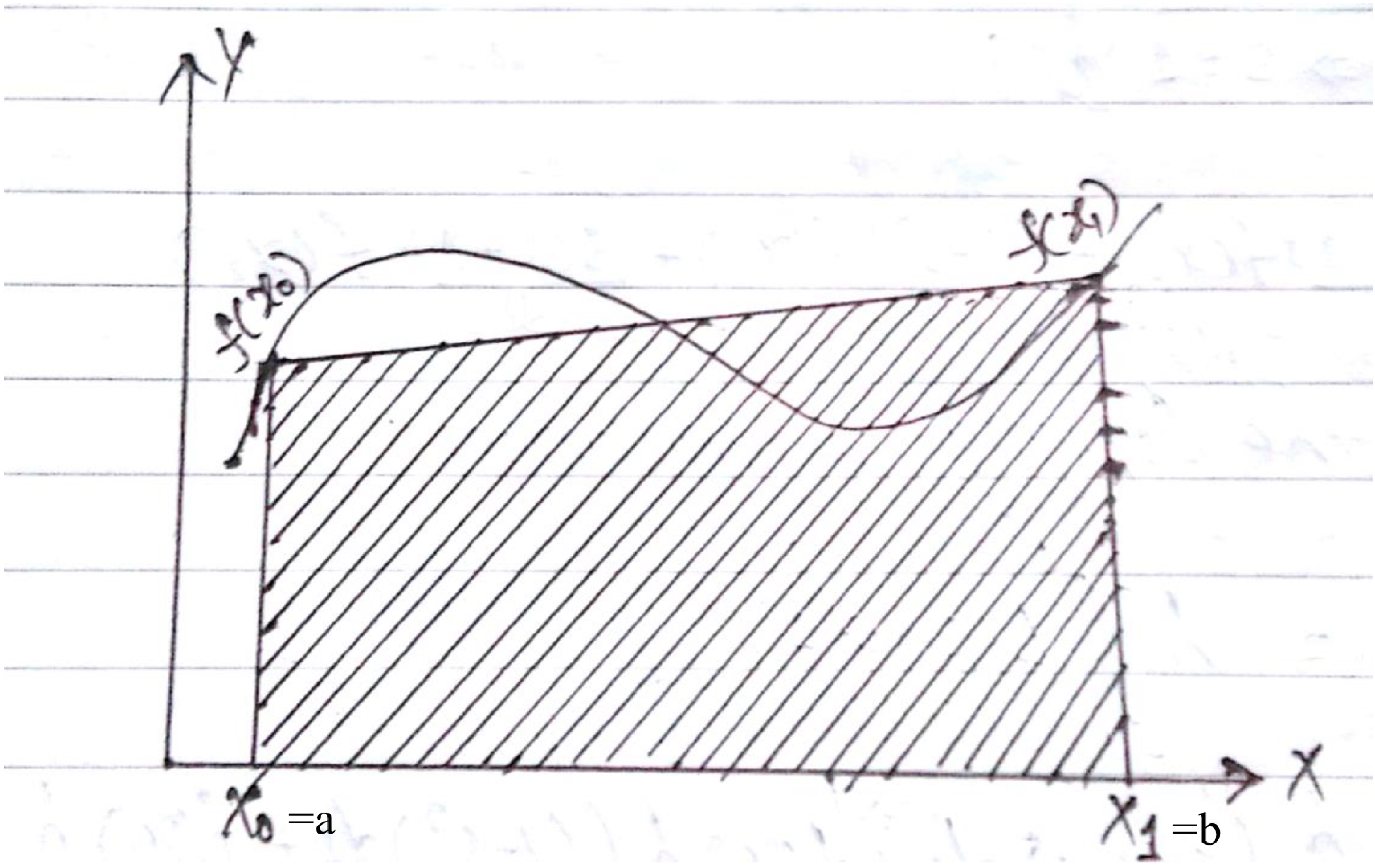
OR

$$\boxed{\int_{x_0}^{x_n} f(x)dx = nh\left[f(x_0) + \frac{n}{2}\Delta f(x_0) + \frac{1}{12}(2n^2 - 3n)\Delta^2 f(x_0) + \frac{1}{24}(n^3 - 4n^2 + 4n)\Delta^3 f(x_0) + \cdots \ldots\right]}$$

This is called **Newton's cotes formula**.

- **<u>Trapezoidal Rule</u>**

Let $f(x)$ be a function and $x_0 = a$ and $x_1 = b$.



$$h = x_1 - x_0$$

As we know from Newton's cotes formula,

$\int_{x_0}^{x_n} f(x)dx = nh\left[f(x_0) + \frac{n}{2}\Delta f(x_0) + \frac{1}{12}(2n^2 - 3n)\Delta^2 f(x_0) + \frac{1}{24}(n^3 - \ 4n^2 + 4n)\Delta^3 f(x_0) + \cdots \ldots \ldots .\,.\right]$ …………….. (i)

By putting $n = 1$ in eq. (i) and neglecting higher term we get,

$\int_{x_0}^{x_1} f(x)dx = h\left[f(x_0) + \frac{1}{2}\Delta f(x_0)\right]$

$= h\left[f(x_0) + \frac{1}{2}[f(x_1) - f(x_0)]\right]$

$= h\left[f(x_0) + \frac{1}{2}f(x_1) - \frac{1}{2}f(x_0)\right]$

$= \frac{h}{2}[f(x_0) + f(x_1)]$

$$\boxed{\therefore \int_{x_0}^{x_1} f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)]}$$

This equation is called *trapezoidal rule.*

Example

Q. Evaluate the integral $\boldsymbol{I = \int_1^2 (x^3 + 1)dx}$***. Using trapezoidal rule.***

Soln:

Here,

$a = 1,\ b = 2,$

We have,

$I = \frac{b-a}{2}[f(a) + f(b)]$

$= \frac{2-1}{2}[2 + 9]$

$= 5.5$

Composite Trapezoidal Rule

If the range is to be integrated is large, the trapezoidal rule can be improved by dividing the interval (a, b) into the small intervals and applying the rule discussed above to each of these subintervals. The sum of the subintervals is the integral of the interval (a, b).

In the fig. below, there are $n + 1$ equally spaced sampling points that creates n segments of equal width h given by,

$$h = \frac{b-a}{n}$$

$$x_i = a + ih, \qquad i = 0, 1, 2, \dots \dots \dots, n$$

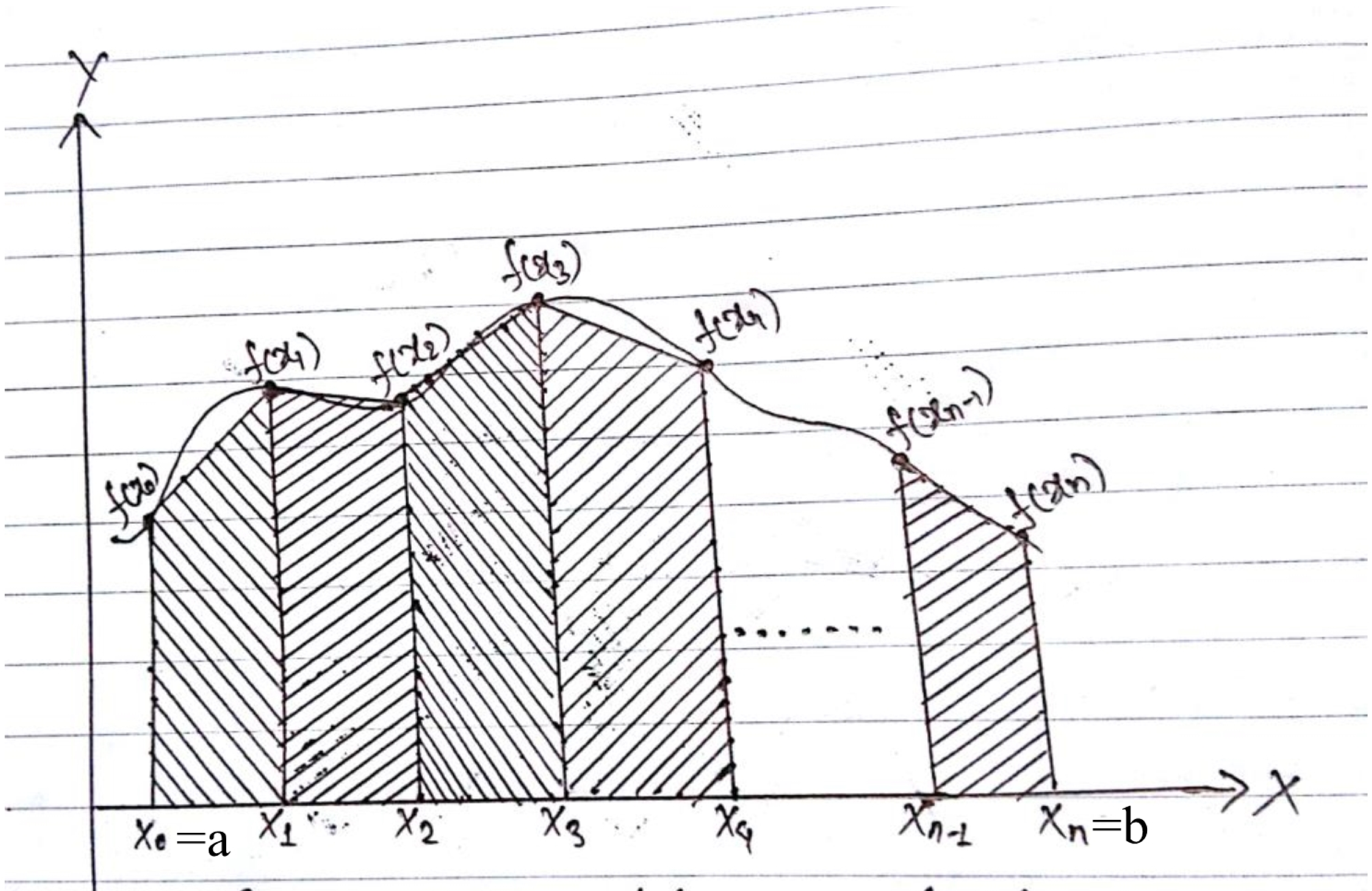


fig: Multisegment trapezoidal rule

From trapezoidal rule, area of the subinterval with the nodes x_{i-1} and x_i is given by

$I = \frac{h}{2}[f(x_{i-1}) + f(x_i)]$

The total area of all the n segments is,

$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{x_3} f(x)dx + \dots \dots \dots .. + \int_{x_{n-1}}^{x_n} f(x)dx$

$= \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] + \frac{h}{2}[f(x_2) + f(x_3)] \dots \dots \dots .. + \frac{h}{2}[f(x_{n-1}) + f(x_n)]$

$= \frac{h}{2}[f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + \cdots \dots .. + f(x_{n-1}) + f(x_n)]$

$= \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots \dots .. + 2f(x_{n-1}) + f(x_n)]$

$= \frac{h}{2}[f(x_0) + 2\sum_i^{n-1} f(x_i) + f(x_n)]$

$$\boxed{\therefore \int_a^b f(x)dx = \frac{h}{2}[f(x_0) + 2\sum_i^{n-1} f(x_i) + f(x_n)]}$$

Which is the *composite trapezoidal rule.*

Algorithm for trapezoidal rule

1. *Start*
2. *Read the value of lower limit of integration (a), upper limit of integration (b) and number of segments (n).*
3. *Compute* $h = (b - a)/n$
   x=a
4. *for i=0 to n*
   $y_i = f(x)$
   $x = x + h$
   Repeat i
5. *compute sum=* y_0+y_n
6. *for i=1 to n-1*
   *sum+=2** y_i
   Repeat i
7. *Compute sum=(h/2)* sum*
8. *Display sum as integral value.*
9. *END*

Examples

1. Compute the integral $\int_0^1 \frac{1}{1+x^2} dx$ ***using trapezoidal rule when n=5.***

Sol:

Here,

$a = 0$, $b = 1$, $n = 5$ so,

$h = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5}$

$f(x) = \frac{1}{1+x^2}$

So we get the following table,

x_i	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$f(x_i)$	1	0.9615	0.8620	0.7352	0.6097	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

By trapezoidal rule; we have

$I = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5]$

$= \frac{1}{10}[1 + 2(0.9615 + 0.8620 + 0.7352 + 0.6097) + 0.5]$

$= 0.78368$

$\therefore \int_0^1 \frac{1}{1+x^2} dx = 0.78368$

2. Evaluate $\int_0^1 e^{-x^2}\, dx$ using trapezoidal rule with n=10.

Sol^n:

Here,

$a = 0$, $b = 1$, $n = 10$ so,

$h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$

$f(x) = e^{-x^2}$

So we get the following table,

x_i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$f(x_i)$	1	0.99	0.96	0.91	0.85	0.77	0.69	0.61	0.52	0.44	0.36

By trapezoidal rule,

$$\int_0^1 e^{-x^2}\, dx = \frac{0.1}{2}[1 + 2(0.99 + 0.96 + 0.91 + 0.85 + 0.77 + 0.69 + 0.61 + 0.52 + 0.44) + 0.36]$$

$\therefore \int_0^1 e^{-x^2}\, dx = 0.74$

3. Evaluate $\int_0^1 \sqrt{sinx + cosx}\, dx$ by trapezoidal rule with h = 0.2.

Sol^n:

Here,

$a = 0$, $b = 1$, $h = 0.2$ so,

$n = \frac{b-a}{h} = \frac{1-0}{0.2} = 5$

$f(x) = \sqrt{sinx + cosx}$

So we get the following table,

x_i	0	0.2	0.4	0.6	0.8	1
$f(x_i)$	1	1.0857	1.448	1.1789	1.1891	1.1755

By trapezoidal rule,

$$\int_0^1 \sqrt{sinx + cosx}\, dx = \frac{0.2}{2}[1 + 2(1.0857 + 1.448 + 1.1789 + 1.1891) + 1.1755]$$

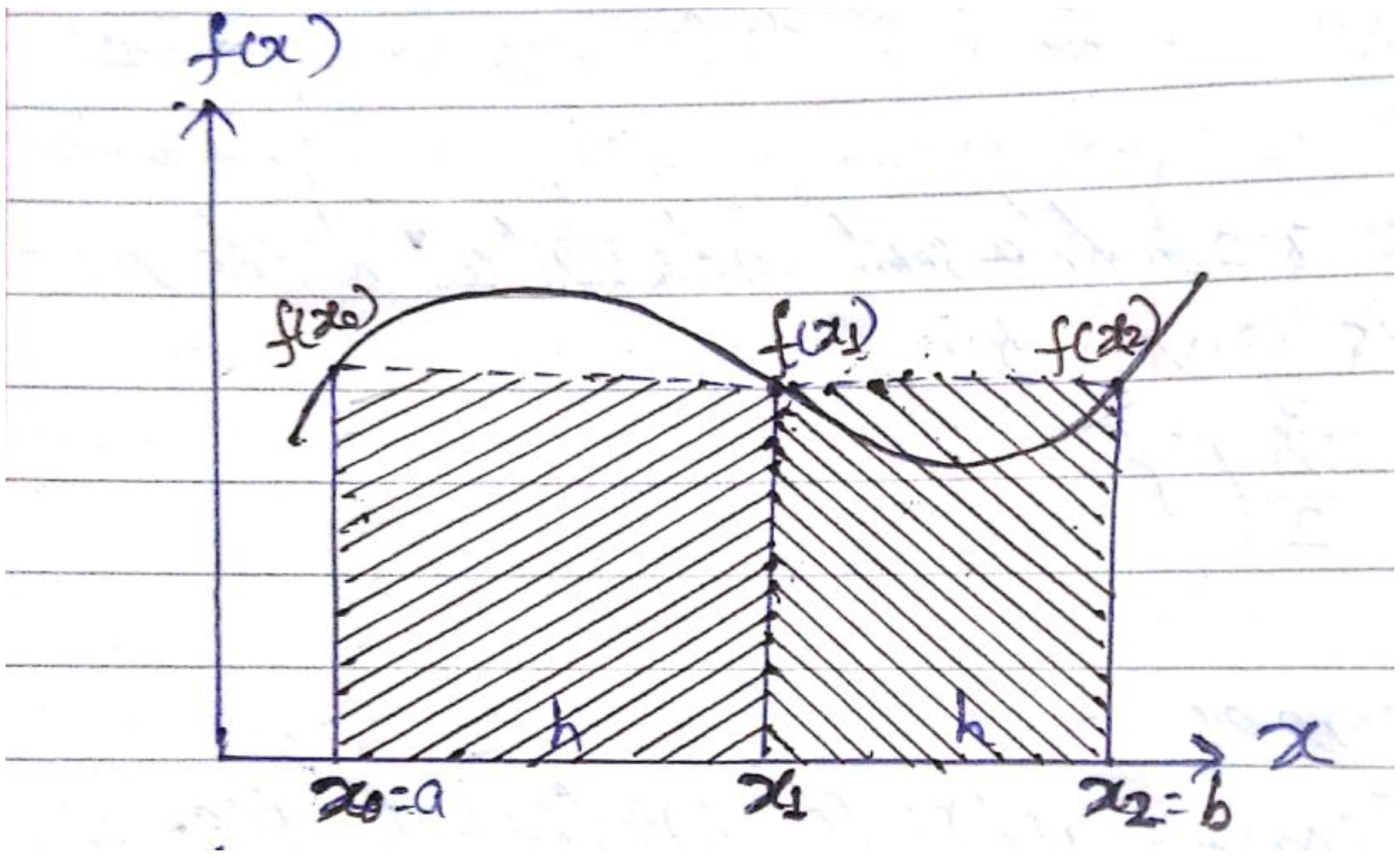
$= 1.19789$

$\therefore \int_0^1 \sqrt{sinx + cosx}\, dx = 1.19789$

Simpson's 1/3 Rule

Here, the function $f(x)$ is approximated by a second-order polynomial $p_2(x)$ which passes through three sampling points as shown in fig. The three points include the end points $x_0(=a)$ and $x_2(=b)$ and a midpoint between them i.e. $x_0 = a$ and $x_2 = b$ and $x_1 = (a+b)/2$. The width of the segments h is given by

$$h = \frac{b-a}{2}$$



As we know from Newton's cotes formula,

$$\int_{x_0}^{x_n} f(x)dx = nh\left[f(x_0) + \frac{n}{2}\Delta f(x_0) + \frac{1}{12}(2n^2 - 3n)\Delta^2 f(x_0) + \frac{1}{24}(n^3 - 4n^2 + 4n)\Delta^3 f(x_0) + \cdots \ldots \ldots\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

By putting $n = 2$ in eq. (i) and neglecting higher term we get,

$$\int_{x_0}^{x_2} f(x)dx = 2h\left[f(x_0) + \Delta f(x_0) + \frac{1}{6}\Delta^2 f(x_0)\right]$$

$$= 2h\left[f(x_0) + [f(x_1) - f(x_0)] + \frac{1}{6}[\Delta f(x_1) - \Delta f(x_0)]\right]$$

$$= 2h\left[f(x_1) + \frac{1}{6}[\{f(x_2) - f(x_1)\} - \{f(x_1) - f(x_0)\}]\right]$$

$$= 2h\left[f(x_1) + \frac{1}{6}[f(x_2) - 2f(x_1) + f(x_0)]\right]$$

$$=2h\left[f(x_1) + \frac{1}{6}f(x_2) - \frac{1}{3}f(x_1) + \frac{1}{6}f(x_0)\right]$$

$$=2h\left[\frac{1}{6}f(x_0) + \frac{2}{3}f(x_1) + \frac{1}{6}f(x_2)\right]$$

$$= \frac{2h}{6}[f(x_0) + 4f(x_1) + f(x_2)]$$

$$= \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

$$\boxed{\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]}$$

This equation is called *Simpson's 1/3 rule.*

Example

***Q.* Evaluate the integral $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx$ using Simpson's 1/3 rule.**

Sol^n^:

Here,

a = 0, b = $\frac{\pi}{2}$ so,

$h = \frac{b-a}{2} = \frac{\frac{\pi}{2}-0}{2} = \frac{\pi}{4}$

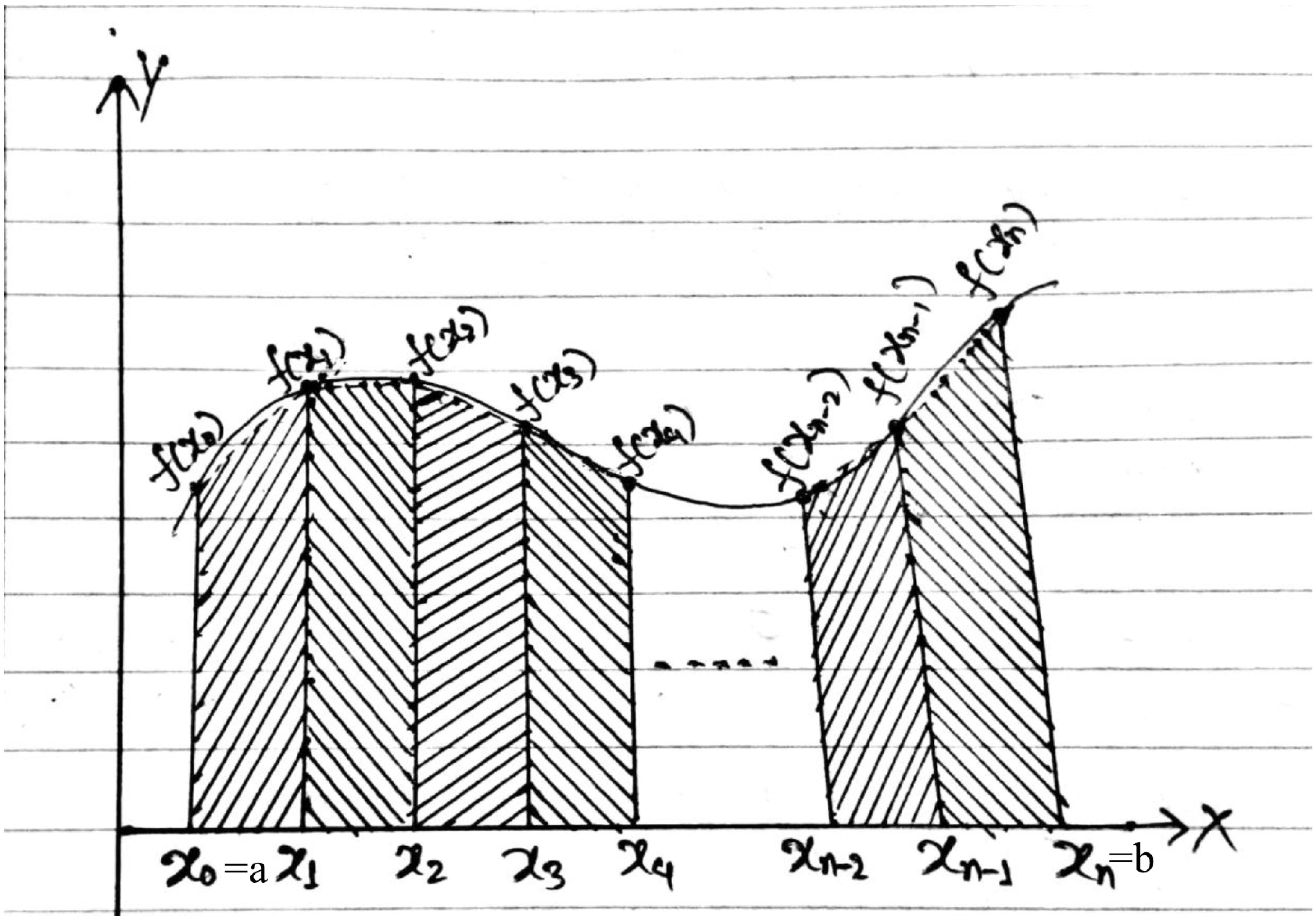
We have, $I = \frac{h}{3}[f(a) + 4f(x_1) + f(b)]$

$x_1 = \frac{a+b}{2} = \frac{0+\frac{\pi}{2}}{2} = \frac{\pi}{4}$

$\therefore\ I = \frac{\pi}{12}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right)\right]$

$= 0.26179(0 + 3.3637 + 1) \quad = \ 1.143$

Composite Simpson's 1/3 Rule



Here, the integration interval is divided into n number of segments of equal width, where n is even number. Then the step size is

$$h = \frac{b-a}{n}$$

$$\int_a^b f(x)dx = \int_a^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots \ldots \ldots \ldots .. + \int_{x_{n-2}}^{b} f(x)dx$$

$$= \frac{h}{3}[f(a) + 4f(x_1) + f(x_2)] + \frac{h}{3}[f(x_2) + 4f(x_3) + f(x_4)] + \ldots \ldots \ldots \ldots \ldots \ldots \ldots + \frac{h}{3}[f(x_{n-2}) + 4f(x_{n-1}) + f(b)]$$

$$= \frac{h}{3}[f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \ldots .. + 4f(x_{2i-1}) + 2f(x_{2i}) + f(b)]$$

$$= \frac{h}{3}\left[f(a) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{\left(\frac{n}{2}\right)-1} f(x_{2i}) + f(b)\right]$$

$$\boxed{\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{\left(\frac{n}{2}\right)-1} f(x_{2i}) + f(b)\right]}$$

This equation is called *composite Simpson's 1/3 rule.*

<u>Algorithm for Simpson's 1/3 rule</u>

1. *Start*
2. *Read the value of lower limit of integration (a), upper limit of integration (b) and number of segments (n).*
3. *Compute* $h = (b - a)/n$
4. *for i=0 to n*
   $x_i = a + i * h$
   $y_i = f(x_i)$
   Repeat i
5. *Initialize so=se=0*
6. *for i=1 to n*
   if (i%2==1)
   so=so+y_i
   else
   se=se+y_i
   Repeat i
7. *ans=(h/3)*(*$y_1 + y_n + 4 * so + 2 * se$*)*
8. *Display ans as integral value.*
9. *END*

<u>Examples</u>

1. Compute the integral $\int_0^1 \frac{1}{1+x^2} dx$ ***applying Simpson's 1/3 rule using n=5.***

<u>Soln:</u>

Here,

a = 0, *b* =1, *n* =5 so,

$h = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5}$

$f(x) = \frac{1}{1+x^2}$

So we get the following table,

x_i	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$f(x_i)$	1	0.9615	0.8620	0.7352	0.6097	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

By Simpson's 1/3 rule; we have,

$I = \frac{h}{3}[y_0 + 2(y_2 + y_4) + 4(y_1 + y_3) + y_5]$

$= \frac{1}{15}[1 + 2(0.8620 + 0.6097) + 4(0.9615 + 0.7352) + 0.5]$

$= 0.74863$

$\therefore \int_0^1 \frac{1}{1+x^2} dx = 0.74863$

2. *Using Simpson's 1/3 rule evaluate* $\int_0^2 (e^{x^2} - 1)\, dx$ ***with*** $n = 8$.

Sol^n:

Here,

$a = 0$, $b = 2$, $n = 8$ so,

$h = \frac{b-a}{n} = \frac{2-0}{8} = 0.25$

$f(x) = e^{x^2} - 1$

So we get the following table,

x_i	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0
$f(x_i)$	0	0.0645	0.2840	0.7550	1.7183	3.7707	8.4877	20.3809	53.5981
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Simpson's 1/3 rule; we have,

$I = \frac{h}{3}[y_0 + 2(y_2 + y_4 + y_6) + 4(y_1 + y_3 + y_5 + y_7) + y_8]$

$= \frac{0.25}{3}[0 + 2(0.2840 + 1.7183 + 8.4877) + 4(0.0645 + 0.7550 + 3.7707 + 20.3809) + 53.5981]$

$= 14.5385$

$\therefore \int_0^2 (e^{x^2} - 1)\, dx = 14.5385$

3. *Using Simpson's 1/3 rule evaluate $\int_{0.2}^{1.2}(x^2 + lnx - sinx)\,dx$. (Take h=0.1)*

<u>Soln:</u>

Here,

a = 0.2, b =1.2, h =0.1

$f(x) = (x^2 + lnx - sinx)$

So we get the following table,

x_i	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
$f(x_i)$	-1.768	-1.409	-1.146	-0.922	-0.715	-0.511	-0.300	-0.079	0.158	0.414	0.690

By Simpson's 1/3 rule; we have,

$\int_{0.2}^{1.2}(x^2 + lnx - sinx)\,dx = \frac{0.1}{3}[-1.768 + 2(-1.146 - 0.715 - 0.300 + 0.158) + 4(-1.409 - 0.922 - 0.511 - 0.079 + 0.414) + 0.690]$

= - 0.5037

$\therefore \int_{0.2}^{1.2}(x^2 + lnx - sinx)\,dx = -0.5037$

- **<u>Simpson's 3/8 Rule</u>**

By putting $n = 3$ in newton's cotes formula and applying the same procedure followed in trapezoidal or Simpson's 1/3 rule, we can show that

$$\boxed{\int_a^b f(x) = \frac{3h}{8}[f(a) + 3f(x_1) + 3f(x_2) + f(b)]}$$

Where, $h = \frac{b-a}{3}$. This equation is called *Simpson's 3/8 rule*.

<u>Example</u>

<u>Q.</u> Use Simpson's 3/8 rule to evaluate $\int_1^2 (x^3 + 1)dx$.

<u>Soln:</u>

Here,

a= 1, b=2

$h = \frac{b-a}{3} = \frac{2-1}{3} = \frac{1}{3}$

$f(x) = (x^3 + 1)$

We have,

$\int_a^b f(x) = \frac{3h}{8}[f(a) + 3f(x_1) + 3f(x_2) + f(b)]$

$x_1 = a + h = 1 + \frac{1}{3} = \frac{4}{3}$

$x_2 = a + 2h = 1 + \frac{2}{3} = \frac{5}{3}$

$\therefore I = \int_1^2 f(x) = \frac{1}{8}[f(1) + 3f(4/3) + 3f(5/3) + f(2)]$

= 4.75

Composite Simpson's 3/8 Rule

$$\int_a^b f(x) = \frac{3h}{8}[f(a) + 3[f(x_1) + f(x_2) + f(x_4) + f(x_5) + \cdots \ldots \ldots . + f(x_{n-1})] + 2[f(x_3) + f(x_6) + \cdots \ldots \ldots + f(x_{n-3})] + f(b)]$$

Examples

1. Integrate the function $\int_0^{\frac{\pi}{2}} \frac{sinx}{x} dx$ by Simpson's 3/8 rule. Take n=5.

Soln:

Here,

a = 0, b = $\frac{\pi}{2}$, n = 5 so,

$h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{5} = \frac{\pi}{10}$

$f(x) = \frac{sinx}{x}$

So we get the following table,

x_i	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{\pi}{2}$
$f(x_i)$	0	0.017453	0.017452	0.017451	0.017451	0.017451
	y_0	y_1	y_2	y_3	y_4	y_5

By Simpson's 3/8 rule;

$\int_0^{\frac{\pi}{2}} \frac{sinx}{x} dx = \frac{3h}{8}[y_0 + 2y_3 + 3(y_1 + y_2 + y_4) + y_5]$

$= \frac{3\pi}{80}[0 + 2 \times 0.017451 + 3(0.017453 + 0.017452 + 0.017451) + 0.017451]$

= 0.02467

2. *Compute the integral $\int_0^6 \frac{1}{1+x^2} dx$ applying Simpson's 3/8 rule using n=6.*

<u>Soln:</u>

Here,

a= 0, b=6 , n=6 so,

$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$

$f(x) = \frac{1}{1+x^2}$

So we get the following table,

x_i	0	1	2	3	4	5	6
$f(x_i)$	1	0.5	0.2	0.1	0.059	0.038	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule;

$\int_0^6 \frac{1}{1+x^2} = \frac{3h}{8}[y_0 + 2y_3 + 3(y_1 + y_2 + y_4 + y_5) + y_6]$

$= \frac{3\text{x}1}{8}[1 + 2y_3 + 3(0.5 + 0.2 + 0.059 + 0.038) + 0.027]$

$= 1.3571$

- **<u>Gaussian Integration</u>**

Gauss integration is based on the concept that the accuracy of numerical integration can be improved by choosing the sampling point rather than on the basis of equal spacing.

Gauss integration assumes an approximation of the form

$$I_g = \int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

Formula:

1. ***<u>For two point</u>***

$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2)$

Where, $w_1 = w_2 = 1$

$x_1 = -\frac{1}{\sqrt{3}}$

$x_2 = \frac{1}{\sqrt{3}}$

$\therefore \int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f(\frac{1}{\sqrt{3}})$

2. ***For three point***

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

Where,

$$w_1 = \frac{5}{9},\ w_2 = \frac{8}{9},\ w_3 = \frac{5}{9}$$

$$x_1 = -\sqrt{\frac{3}{5}}$$

$$x_2 = 0$$

$$x_3 = \sqrt{\frac{3}{5}}$$

Examples

1. Evaluate $\int_{-1}^{1} \frac{1}{1+x^2} dx$ using gauss integration 2 point and 3 point formula.

Sol^n:

Given,

$f(x) = \frac{1}{1+x^2}$

By using 2 point Gaussian formula:

We have, from two point gaussian formula;

$\int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

$\therefore f\left(-\frac{1}{\sqrt{3}}\right) = \frac{1}{1+\left(-\frac{1}{\sqrt{3}}\right)^2} = 0.75$

$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = 0.75$

$\therefore \int_{-1}^{1} \frac{1}{1+x^2} dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.75 + 0.75 = 1.5$

By using 3 point Gaussian formula:

We have, from 3 point gaussian formula;

$\int_{-1}^{1} f(x)dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$

$\therefore \int_{-1}^{1} \frac{1}{1+x^2} dx = \frac{5}{9}(0.625) + \frac{8}{9}(1) + \frac{5}{9}(0.625) = 1.5833$

2. *Using two point gauss Legendre formula evaluate $\int_{0.2}^{1.5} e^{-x^2}\,dx$.*

Soln:

Given that,

$I = \int_{0.2}^{1.5} e^{-x^2}\,dx$

Changing limit from (0.2, 1.5) to (-1, 1) by transformation,

$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$

$= \frac{1}{2}(1.5-0.2)u + \frac{1}{2}(1.5+0.2)$

$=0.65u + 0.85$

$\frac{dx}{du} = 0.65 \Rightarrow dx = 0.65\,du$

Now,

$\int_{0.2}^{1.5} e^{-x^2}\,dx = \int_{-1}^{1} e^{-(0.65u+0.85)^2} \times 0.65\,du$

$\therefore \quad f(u) = 0.65e^{-(0.65u+0.85)^2}$

$f\left(-\frac{1}{\sqrt{3}}\right) = 0.5044$

$f\left(\frac{1}{\sqrt{3}}\right) = 0.1553$

We have, from two point gauss Legendre formula;

$I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.5044 + 0.1553 = 0.6597$

3. *Evaluate $\int_1^2 (lnx + x^2 sinx)dx$ using Gauss integration 3 point formula.*

Soln:

Given that,

$I = \int_1^2 (lnx + x^2 sinx)dx$

Changing limit from (1, 2) to (-1, 1) by transformation,

$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$

$= \frac{1}{2}(2-1)u + \frac{1}{2}(2+1)$

$=0.5u + 1.5$

$\frac{dx}{du} = 0.5 \Rightarrow dx = 0.5\,du$

Now,

$$\int_1^2 (lnx + x^2 sinx)dx = \int_{-1}^1 [\ln(0.5u + 1.5) + (0.5u + 1.5)^2 \sin(0.5u + 1.5)] \times 0.5\, du$$

$\therefore \quad f(u) = 0.5\,[\ln(0.5u + 1.5) + (0.5u + 1.5)^2 \sin(0.5u + 1.5)]$

$$f\left(-\sqrt{\frac{3}{5}}\right) = 0.06542$$

$$f(0) = 0.23218$$

$$f\left(\sqrt{\frac{3}{5}}\right) = 0.37623$$

By Gauss 3-point formula,

$I = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$

$$= \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$$= \frac{5}{9} \times 0.06542 + \frac{8}{9} \times 0.23218 + \frac{5}{9} \times 0.37623$$

$$= 0.45174$$

4. *Evaluate $\int_{.5}^{1.5} e^{-x}\, dx$ using the Gaussian integration three point formula.*

Sol^n^:

Given that,

$I = \int_{.5}^{1.5} e^{-x}\, dx$

Changing limit from (0.5, 1.5) to (-1, 1) by transformation,

$x = \frac{1}{2}(b - a)u + \frac{1}{2}(b + a)$

$$= \frac{1}{2}(1.5 - 0.5)u + \frac{1}{2}(1.5 + 0.5)$$

$$= 0.5u + 1$$

$\frac{dx}{du} = 0.5 \Rightarrow dx = 0.5\, du$

Now,

$\int_{.5}^{1.5} e^{-x}\, dx = \int_{-1}^{1} e^{-(0.5u+1)} \times 0.5\, du$

$\therefore \quad f(u) = 0.5e^{-(0.5u+1)}$

So,

$$f\left(-\sqrt{\frac{3}{5}}\right) = 0.2709$$

$$f(0) = 0.1839$$

$$f\left(\sqrt{\frac{3}{5}}\right) = 0.1249$$

By Gauss 3-point formula,

$$I = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

$$= \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$$= \frac{5}{9} \times 0.2709 + \frac{8}{9} \times 0.1839 + \frac{5}{9} \times 0.1249$$

$$= 0.38336$$

- **Romberg Integration**

The Romberg integration method uses the trapezoidal rule. It use trapezoidal rule in iterative way.

Formula to find Romberg integration

Use trapezoidal rule to find

I_1 = Divide the interval into two parts

I_2 = Divide the interval into four parts

I_3 = Divide the interval into eight parts

Then find,

$$I_4 = \frac{4I_2 - I_1}{3}$$

$$I_5 = \frac{4I_3 - I_2}{3}$$

Now, final result

$$I = \frac{4I_5 - I_4}{3}$$

Examples

1. *Use Romberg integration method to evaluate integration $\int_4^{5.2} \log x\, dx$.*

Sol^n:

We have $h = \frac{b-a}{2} = \frac{5.2-4}{2} = 0.6$

Therefore, taking $h = 0.6$, $\frac{0.6}{2} = 0.3$ & $\frac{0.3}{2} = 0.15$

Let us calculate the given integral using trapezoidal rule.

$$f(x) = \log x$$

i) Taking $h = 0.6$

x_i	4	4.6	5.2
$f(x_i)$	0.60206	0.662758	0.716003

$I_1 = \frac{0.6}{2}[0.60206 + 2 \times 0.662758 + 0.716003] = 0.793$

ii) Taking $h = 0.3$

x_i	4	4.3	4.6	4.9	5.2
$f(x_i)$	0.60206	0.633468	0.662758	0.690196	0.716003

$I_2 = \frac{0.3}{2}[0.60206 + 2(0.633468 + 0.662758 + 0.690196) + 0.716003] = 0.794$

iii) Taking $h = 0.15$

x_i	4	4.15	4.3	4.45	4.6	4.75	4.9	5.04	5.2
$f(x_i)$	0.60206	0.618048	0.633468	0.64836	0.662758	0.67669	0.690196	0.70243	0.716003

$I_3 = \frac{0.15}{2}[0.60206 + 2(0.618048 + 0.633468 + 0.64836 + 0.662758 + 0.67669 + 0.70243 + 0.690196) + 0.716003] = 0.793$

Now,

$I_4 = \frac{4I_2 - I_1}{3} = \frac{4 \times 0.794 - 0.793}{3} = 0.794$

$I_5 = \frac{4I_3 - I_2}{3} = \frac{4 \times 0.793 - 0.794}{3} = 0.792$

$I = \frac{4I_5 - I_4}{3} = \frac{4 \times 0.792 - 0.794}{3} = 0.791$

$\therefore \int_4^{5.2} \boldsymbol{\log x\, dx = 0.791}$

2. *Use Romberg integration method to evaluate integration* $\int_0^1 \frac{1}{1+x}\,dx$.

Soln:

We have $h = \frac{b-a}{2} = \frac{1-0}{2} = 0.5$

Therefore, taking $h = 0.5$, $\frac{0.5}{2} = 0.25$ & $\frac{0.25}{2} = 0.125$

Let us calculate the given integral using trapezoidal rule.

$$f(x) = \frac{1}{1+x}$$

i) Taking $h = 0.5$

x_i	0	0.5	1
$f(x_i)$	1	0.667	0.5

$I_1 = \frac{0.5}{2}[1 + 2 \times 0.667 + 0.5] = 0.708$

ii) Taking $h = 0.25$

x_i	0	0.25	0.5	0.75	1
$f(x_i)$	1	0.8	0.667	0.5714	0.5

$I_2 = \frac{0.25}{2}[1 + 2(0.8 + 0.667 + 0.5714) + 0.5] = 0.696$

iii) Taking $h = 0.125$

x_i	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$f(x_i)$	1	0.889	0.8	0.7273	0.667	0.615	0.5714	0.533	0.5

$I_3 = \frac{0.125}{2}[1 + 2(0.889 + 0.8 + 0.7273 + 0.667 + 0.615 + 0.5714 + 0.533) + 0.5] =$
0.6941

Now,

$I_4 = \frac{4I_2 - I_1}{3} = \frac{4 \times 0.696 - 0.708}{3} = 0.693$

$I_5 = \frac{4I_3 - I_2}{3} = \frac{4 \times 0.6941 \; - 0.696}{3} = 0.693$

$I = \frac{4I_5 - I_4}{3} = \frac{4 \times 0.693 \; - 0.693}{3} = 0.693$

$\therefore \int_0^1 \frac{1}{1+x}\,dx = \mathbf{0.693}$

References:

- *E. Balagurusamy, Numerical Methods, Tata McGraw-Hill*

Please let me know if I missed anything or anything is incorrect.

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