Unit 4

Solution of Linear Algebraic Equations

A system of n linear equations is represented generally as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

The techniques for solving system of linear algebraic equation are;

Eliminative technique: In this technique system is reduced to a form from which the solution can be obtained by direct or simple substitution. Some eliminative techniques are:

- Gauss elimination method
- Gauss-Jordan method
- LU decomposition method
- Matrix inversion method

Iterative technique: This technique involves assumption of some initial values which are then refined repeatedly till they reach some accepted level of accuracy. Some iterative techniques are:

- Jacobi iteration method
- Gauss seidal iteration method

Ill-conditioned system

Systems where small changes in the coefficient of variable shows large deviation in the solution are said to be ill-conditioned systems. For e.g.

$2x_1 + x_2 = 25$………. (i)

$2.001x_1 + x_2 = 25.01$………. (ii)

Solving eq. (i) & (ii) we get

$x_1 = 10$ & $x_2 = 5$

Let us change the coefficient of x_1 in eq. (ii) to 2.0005. Now the values of x_1 & x_2 are

$x_1 = 20$ & $x_2 = -15$

Here, small change in the coefficient of variable showed large change in the solution. So such systems are called ill-conditioned system.

➢ Gauss Elimination Method

Gauss elimination method proposes a systematic strategy for reducing the system of equation to upper triangular form using forward elimination approach and then for obtaining values of unknowns using the back substitution process.

Consider a system of linear equation;

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

The augmented matrix of given system is;

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}: b_1 \\ a_{21} & a_{22} & a_{23}: b_2 \\ a_{31} & a_{32} & a_{33}: b_3 \end{bmatrix}$$

Doing the row operation, the augmented matrix reduces to upper triangular matrix as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}: & b_1 \\ 0 & a'_{22} & a'_{23}: & b'_2 \\ 0 & 0 & a''_{33}: & b''_3 \end{bmatrix}$$

Now, evaluate the unknown values using backward substitution

$a''_{33}x_3 = b''_3$

or, $x_3 = \frac{b''_3}{a''_{33}}$

$a'_{22}x_2 + a'_{23}x_3 = b'_2$

or, $x_2 = \frac{b'_2 - a'_{23}x_3}{a'_{22}}$

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

or, $x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$

Examples

1. Solve the following set of equations using Gauss elimination method.

$$\boldsymbol{x_1 + 10x_2 + x_3 = 24}$$
$$\boldsymbol{10x_1 + x_2 + x_3 = 15}$$
$$\boldsymbol{x_1 + x_2 + 10x_3 = 33}$$

Sol^n:

The augmented matrix of given system is;

$$\begin{bmatrix} 1 & 10 & 1 & : 24 \\ 10 & 1 & 1 & : 15 \\ 1 & 1 & 10 & : 33 \end{bmatrix}$$

$Applying, R_2 \rightarrow R_2 - 10R_1 , R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 10 & 1 & : & 24 \\ 0 & -99 & -9 & : & -225 \\ 0 & -9 & 9 & : & 9 \end{bmatrix}$$

$Applying, R_3 \rightarrow R_3 - \frac{1}{11}R_2$

$$\begin{bmatrix} 1 & 10 & 1 & : & 24 \\ 0 & -99 & -9 & : & -225 \\ 0 & 0 & \frac{108}{11} & : & \frac{324}{11} \end{bmatrix}$$

Now, using backward substitution;

$\frac{108}{11}x_3 = \frac{324}{11}$

or, $x_3 = 3$

$-99x_2 - 9x_3 = -225$

or, $x_2 = \frac{-225+9\times3}{-99} = 2$

$x_1 + 10x_2 + x_3 = 24$

or, $x_1 = \frac{24-10\times2-3}{1} = 1$

Hence,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Which is the required solution of the system.

2. ***Use Gauss Elimination to solve the following system of equation.***

2x+3y+4z=5
3x+4y+5z=6
4x+5y+6z=7.

Sol^n:

Given system of equation,

2x+3y+4z=5
3x+4y+5z=6
4x+5y+6z=7

The augmented matrix of given system is;

$$\begin{bmatrix} 2 & 3 & 4 & : 5 \\ 3 & 4 & 5 & : 6 \\ 4 & 5 & 6 : & 7 \end{bmatrix}$$

$Applying, R_2 \to R_2 - \frac{3}{2}R_1 \,, R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 2 & 3 & 4: & 5 \\ 0 & -1/2 & -1: & -3/2 \\ 0 & -1 & -2: & -3 \end{bmatrix}$$

$Applying, R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 2 & 3 & 4: & 5 \\ 0 & -1/2 & -1: & -3/2 \\ 0 & 0 & 0: & 0 \end{bmatrix}$$

Here, pivot column are 1st and 2nd, so basic variable are x and y and z is free variable. So, the given system has non-trivial solution.

3. ***Solve the following set of linear equations using Gauss Elimination method.***

$$\mathbf{2x_2 + x_4 = 0}$$
$$\mathbf{2x_1 + 2x_2 + 3x_3 + 2x_4 = -2}$$
$$\mathbf{4x_1 - 3x_2 + x_4 = -7}$$
$$\mathbf{6x_1 + x_2 - 6x_3 - 5x_4 = 6}$$

Sol^n:

Given system of equation,

$$2x_2 + x_4 = 0$$
$$2x_1 + 2x_2 + 3x_3 + 2x_4 = -2$$
$$4x_1 - 3x_2 + x_4 = -7$$
$$6x_1 + x_2 - 6x_3 - 5x_4 = 6$$

The augmented matrix of given system is;

$$\begin{bmatrix} 0 & 2 & 0 & 1 & : & 0 \\ 2 & 2 & 3 & 2 & : & -2 \\ 4 & -3 & 0 & 1 & : & -7 \\ 6 & 1 & -6 & -5 & : & 6 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & 2 & 3 & 2 & : & -2 \\ 0 & 2 & 0 & 1 & : & 0 \\ 4 & -3 & 0 & 1 & : & -7 \\ 6 & 1 & -6 & -5 & : & 6 \end{bmatrix}$$

Applying, $R_3 \to R_3 - 2R_1, R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 2 & 2 & 3 & 2 & : & -2 \\ 0 & 2 & 0 & 1 & : & 0 \\ 0 & -7 & -6 & -3 & : & -3 \\ 0 & -5 & -15 & -11 & : & 12 \end{bmatrix}$$

Applying, $R_3 \rightarrow R_3 + \frac{7}{2}R_2, R_4 \rightarrow R_4 + \frac{5}{2}R_2$

$$\begin{bmatrix} 2 & 2 & 3 & 2 & : & -2 \\ 0 & 2 & 0 & 1 & : & 0 \\ 0 & 0 & -6 & \frac{1}{2} & : & -3 \\ 0 & 0 & -15 & -\frac{17}{2} & : & 12 \end{bmatrix}$$

Applying, $R_4 \rightarrow R_4 - \frac{15}{6}R_3$

$$\begin{bmatrix} 2 & 2 & 3 & 2 & : & -2 \\ 0 & 2 & 0 & 1 & : & 0 \\ 0 & 0 & -6 & \frac{1}{2} & : & -3 \\ 0 & 0 & 0 & -\frac{39}{4} & : & \frac{39}{2} \end{bmatrix}$$

Now, using backward substitution;

$-\frac{39}{4}x_4 = \frac{39}{2} \quad \Rightarrow x_4 = -2$

$-6x_3 + \frac{1}{2}x_4 = -3 \quad \Rightarrow \quad x_3 = \frac{1}{3}$

$2x_2 + x_4 = 0 \quad \Rightarrow \quad x_2 = 1$

$2x_1 + 2x_2 + 3x_3 + 2x_4 = -2 \quad \Rightarrow \quad x_1 = -\frac{1}{2}$

$\therefore \; \boldsymbol{x_1 = -\frac{1}{2}, x_2 = 1, x_3 = \frac{1}{3}, x_4 = -2}$

Pivoting

Pivoting is the process of recording the equation in order to get better accuracy. Pivoting can be done only if the pivot element is non-zero. The main diagonal element of augmented matrix is called pivot element.

Gauss elimination with partial pivoting

The partial pivoting involves the following steps;

1. Search and locate the largest absolute value among the coefficients in the first column.
2. Exchange the first row with the row containing that element.
3. Then eliminate the first variable in the second equation.
4. When second row becomes the pivot row, search for the coefficients in the second column from the second row to the n^{th} row and locate the largest coefficient.
5. Exchange the second row with the row containing largest coefficient.
6. Continue the process until (n-1) unknowns are eliminated.

Complete Pivoting

In order to get more accuracy we can also search the whole matrix and bring the largest element in magnitude as the pivot element by interchanging rows as well as column. This is called complete pivoting.

Example

Q. Using Gauss elimination with partial pivoting, solve the following sets of equations.

$$\mathbf{2x_1 + 2x_2 + x_3 = 6}$$
$$\mathbf{4x_1 + 2x_2 + 3x_3 = 4}$$
$$\mathbf{x_1 - x_2 + x_3 = 0}$$

Sol^n:

The augmented matrix of given system is;

$$\begin{bmatrix} 2 & 2 & 1 & : & 6 \\ 4 & 2 & 3 & : & 4 \\ 1 & -1 & 1 & : & 0 \end{bmatrix}$$

Check the first column for largest coefficient. Here, R_2 is pivot equation. (4 is the largest coefficient)

$R_1 \leftrightarrow R_2$; We have

$$\begin{bmatrix} 4 & 2 & 3 & : & 4 \\ 2 & 2 & 1 & : & 6 \\ 1 & -1 & 1 & : & 0 \end{bmatrix}$$

$$Applying, R_2 \to R_2 - \frac{1}{2}R_1 \text{ , } R_3 \to R_3 - \frac{1}{4}R_1$$

$$\begin{bmatrix} 4 & 2 & 3 & : & 4 \\ 0 & 1 & -0.5 & : & 4 \\ 0 & -1.5 & 0.25 & : & -1 \end{bmatrix}$$

Now, check the largest value for second column except first row. Here, -1.5 is the largest coefficient i.e. R_2 is the pivot equation. [*Note: Absolute value of -1.5 is greater than 1*]

$R_2 \leftrightarrow R_3$; We have;

$$\begin{bmatrix} 4 & 2 & 3 & : & 4 \\ 0 & -1.5 & 0.25 & : & -1 \\ 0 & 1 & -0.5 & : & 4 \end{bmatrix}$$

$Applying, R_3 \rightarrow R_3 + \frac{2}{3}R_2$

$$\begin{bmatrix} 4 & 2 & 3 & : & 4 \\ 0 & -1.5 & 0.25 & : & -1 \\ 0 & 0 & -0.33 & : & 3.33 \end{bmatrix}$$

Using backward substitution;

$-0.33x_3 = 3.33$

$\therefore\ x_3 = -\frac{3.33}{0.33} = -10$

$-1.5x_2 + 0.25x_3 = -1$

$\therefore\ x_2 = \frac{-1+0.25\times10}{-1.5} = -1$

$4x_1 + 2x_2 + 3x_3 = 4$

$\therefore\ x_1 = \frac{4+2\times1+3\times10}{4} = 9$

Hence,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \\ -1 \end{bmatrix}$$

Which is the required solution of the system.

- **Gauss Jordan Method**

In this method, the augmented matrix is reduced to unit diagonal matrix. Here, we get the solution without using back substitution.

Consider a system of linear equation;

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

The augmented matrix of given system is;

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}: & b_1 \\ a_{21} & a_{22} & a_{23}: & b_2 \\ a_{31} & a_{32} & a_{33}: & b_3 \end{bmatrix}$$

Doing the row operation, the augmented matrix reduces to unit diagonal matrix as;

$$\begin{bmatrix} 1 & 0 & 0: & b'_1 \\ 0 & 1 & 0: & b''_2 \\ 0 & 0 & 1: & b'''_3 \end{bmatrix}$$

Hence, the required solution is;

$$x_1 = b'_1$$
$$x_2 = b''_2$$
$$x_3 = b'''_3$$

Examples

1. Solve the following system of linear equation using Gauss Jordan method.

$$\mathbf{4x_1 + 3x_2 - x_3 = 6}$$
$$\mathbf{x_1 + x_2 + x_3 = 1}$$
$$\mathbf{3x_1 + 5x_2 + 3x_3 = 4}$$

Soln:

The augmented matrix of given system is;

$$\begin{bmatrix} 4 & 3 & -1 & : 6 \\ 1 & 1 & 1 & : 1 \\ 3 & 5 & 3: & 4 \end{bmatrix}$$

$Applying, R_2 \rightarrow R_2 - \frac{1}{4}R_1 , R_3 \rightarrow R_3 - \frac{3}{4}R_1$

$$\begin{bmatrix} 4 & 3 & -1 & : & 6 \\ 0 & 0.25 & 1.25 & : & -0.5 \\ 0 & 2.75 & 3.75: & & -0.5 \end{bmatrix}$$

$Applying, R_3 \rightarrow R_3 - 11R_2$

$$\begin{bmatrix} 4 & 3 & -1 & : & 6 \\ 0 & 0.25 & 1.25 & : & -0.5 \\ 0 & 0 & -10: & & 5 \end{bmatrix}$$

$Applying, R_1 \rightarrow R_1 - 12R_2$

$$\begin{bmatrix} 4 & 0 & -16 & : & 12 \\ 0 & 0.25 & 1.25 & : & -0.5 \\ 0 & 0 & -10: & & 5 \end{bmatrix}$$

$Applying, R_1 \rightarrow R_1 - \frac{8}{5}R_3 , R_2 \rightarrow R_2 + \frac{1}{8}R_3$

$$\begin{bmatrix} 4 & 0 & 0 & : & 4 \\ 0 & 0.25 & 0 & : & 0.125 \\ 0 & 0 & -10: & & 5 \end{bmatrix}$$

$Applying, R_1 \to \frac{1}{4}R_1, \quad R_2 \to \frac{1}{0.25}R_2, \quad R_3 \to \frac{1}{-10}R_3$

$$\begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 0.5 \\ 0 & 0 & 1 & : & -0.5 \end{bmatrix}$$

The equation can be written as

$x_1 = 1$

$x_2 = 0.5$

$x_3 = -0.5$

Which is the required solution of the system.

2. Solve the following system of linear equation using Gauss Jordan method.

$$\begin{aligned} x_1 + x_2 + x_3 &= 9 \\ 2x_1 - 3x_2 + 4x_3 &= 13 \\ 3x_1 + 4x_2 + 5x_3 &= 40 \end{aligned}$$

Sol^n:

The augmented matrix of given system is;

$$\begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & -3 & 4 & : & 13 \\ 3 & 4 & 5 & : & 40 \end{bmatrix}$$

$Applying, R_2 \to R_2 - 2R_1 \, , R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -5 & 2 & : & -5 \\ 0 & 1 & 2 & : & 13 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 1 & 2 & : & 13 \\ 0 & -5 & 2 & : & -5 \end{bmatrix}$$

$Applying, R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 12 & : & 60 \end{bmatrix}$$

$Applying, R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & -1 & : & -4 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 12 & : & 60 \end{bmatrix}$$

$Applying, R_1 \to R_1 + \frac{1}{12}R_3 \, , R_2 \to R_2 - \frac{1}{6}R_3$

$$\begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 12 & : & 60 \end{bmatrix}$$

$Applying, \quad R_3 \rightarrow \frac{1}{12} R_3$

$$\begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

This gives,

$x_1 = 1$

$x_2 = 3$

$x_3 = 5$

Which is the required solution of the system.

Q. Compare Gauss Elimination method and Gauss Jordan method of solving simultaneous equation.

Solution:

In Gauss- elimination method, the augmented matrix is reduced to Echelon form using row operations and then back substitution is applied to get values of unknowns, occurring in a system of linear equations.

In Gauss-Jordan method, the augmented matrix is reduced to reduced Echelon form using elementary row operations to obtain values of unknowns of a system of linear equations.

- **Triangular Factorization Method**

This method is based on the fact that every square matrix 'A' can be decomposed into lower triangular matrix (L) and upper triangular matrix (U) such that **A=LU**.

Consider a system of linear equation;

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

This system of equation can be written in the matrix form as;

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Or, $AX = B$ .... (i)

Where, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Let, $A = LU$

Where, $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ & U= $\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$

Now, eq. (i) becomes,

$LUX = B$

Or, $LZ = B$

Where, $Z = UX$

Now, we can solve the equation in two stages:

i) Solve the equation LZ=B for 'z' by forward substitution.

ii) Solve the equation UX=Z for 'x' using 'z' by backward substitution.

Doolittle and Cholesky Factorization

The factorization is done by assuming diagonal element of 'L' and 'U' to be unity.

Doolittle Factorization:

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = A$$

Cholesky Factorization:

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix} = A$$

Example

Q. ***Solve the following system using Dolittle LU decomposition method.***

$$\mathbf{3x_1 + 2x_2 + x_3 = 10}$$

$$\mathbf{2x_1 + 3x_2 + 2x_3 = 14}$$

$$\mathbf{x_1 + 2x_2 + 3x_3 = 14}$$

Soln:

The given system of linear equation can be written in matrix form as;

$$\begin{bmatrix}3 & 2 & 1\\2 & 3 & 2\\1 & 2 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}10\\14\\14\end{bmatrix}$$

or, AX=B

where, $A=\begin{bmatrix}3 & 2 & 1\\2 & 3 & 2\\1 & 2 & 3\end{bmatrix}$, X= $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ & $B=\begin{bmatrix}10\\14\\14\end{bmatrix}$

Let A=LU

where, $L=\begin{bmatrix}1 & 0 & 0\\l_{21} & 1 & 0\\l_{31} & l_{32} & 1\end{bmatrix}$ & U= $\begin{bmatrix}U_{11} & U_{12} & U_{13}\\0 & U_{22} & U_{23}\\0 & 0 & U_{33}\end{bmatrix}$

$$\therefore \begin{bmatrix}1 & 0 & 0\\l_{21} & 1 & 0\\l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}U_{11} & U_{12} & U_{13}\\0 & U_{22} & U_{23}\\0 & 0 & U_{33}\end{bmatrix}=\begin{bmatrix}3 & 2 & 1\\2 & 3 & 2\\1 & 2 & 3\end{bmatrix}$$

or, $\begin{bmatrix}U_{11} & U_{12} & U_{13}\\l_{21}U_{11} & l_{21}U_{12}+U_{22} & l_{21}U_{13}+U_{23}\\l_{31}U_{11} & l_{31}U_{12}+l_{32}U_{22} & l_{31}U_{13}+l_{32}U_{23}+U_{33}\end{bmatrix}=\begin{bmatrix}3 & 2 & 1\\2 & 3 & 2\\1 & 2 & 3\end{bmatrix}$

Equating corresponding element, we get;

$\boldsymbol{U_{11}=3}$, $\boldsymbol{U_{12}=2}$, $\boldsymbol{U_{13}=1}$

$l_{21}U_{11}=2 \Rightarrow \boldsymbol{l_{21}=\frac{2}{3}}$, $l_{21}U_{12}+U_{22}=3 \Rightarrow \boldsymbol{U_{22}=\frac{5}{3}}$

$l_{21}U_{13}+U_{23}=2 \Rightarrow \boldsymbol{U_{23}=\frac{4}{3}}$, $l_{31}U_{11}=1 \Rightarrow \boldsymbol{l_{31}=\frac{1}{3}}$

$l_{31}U_{12}+l_{32}U_{22}=2 \Rightarrow \boldsymbol{l_{32}=\frac{4}{5}}$, $l_{31}U_{13}+l_{32}U_{23}+U_{33}=3 \Rightarrow \boldsymbol{U_{33}=\frac{8}{5}}$

Hence,

$L=\begin{bmatrix}1 & 0 & 0\\\frac{2}{3} & 1 & 0\\\frac{1}{3} & \frac{4}{5} & 1\end{bmatrix}$ & $U=\begin{bmatrix}3 & 2 & 1\\0 & \frac{5}{3} & \frac{4}{3}\\0 & 0 & \frac{8}{5}\end{bmatrix}$

We know that,

LZ=B

$$\begin{bmatrix}1 & 0 & 0\\\frac{2}{3} & 1 & 0\\\frac{1}{3} & \frac{4}{5} & 1\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}=\begin{bmatrix}10\\14\\14\end{bmatrix}$$

Using forward substitution we get,

$z_1 = 10$

$z_2 = \frac{22}{3}$

$z_3 = \frac{24}{5}$

Again we have,

UX=Z

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & \frac{8}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ \frac{22}{3} \\ \frac{24}{5} \end{bmatrix}$$

Using backward substitution we get,

$x_3 = 3$

$x_2 = 2$

$x_1 = 1$

- **<u>Matrix Inversion Method</u>**

Gauss Jordan elimination method can be used to find the inverse of a matrix. Consider the matrix A of order 3×3.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

This is done as follows:

1. Augment the coefficient matrix A with an identity matrix as shown below:

$$[A \quad I] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix}$$

2. Apply the Gauss-Jordan method to the augmented matrix to reduce A to an identity matrix. The result will be as shown below;

$$\begin{bmatrix} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{bmatrix} = [I \quad A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Example

Q. *Determine the inverse matrix of the given matrix.*

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \\ \mathbf{4} & \mathbf{1} & \mathbf{8} \end{bmatrix}$$

Sol^n:

Consider the matrix

$[A \quad I]$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 4R_1$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & -1 & -1 & -2 & 1 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{bmatrix}$$

$R_3 \rightarrow -R_3$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{bmatrix}$$

Therefore we get;

$$\boldsymbol{A^{-1}} = \begin{bmatrix} \mathbf{-11} & \mathbf{2} & \mathbf{2} \\ \mathbf{-4} & \mathbf{0} & \mathbf{1} \\ \mathbf{6} & \mathbf{-1} & \mathbf{-1} \end{bmatrix}$$

Gauss Jacobi Iterative Method

Consider a system of linear equation;

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

Then, these equation can be written as;

$$\left.\begin{aligned} x_1 &= \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}} \\ x_2 &= \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}} \\ x_3 &= \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}} \end{aligned}\right\} \text{------------(i)}$$

Here, a_{11}, a_{22}, a_{33} must be larger than other coefficient in each equation. (i.e. the given system must be diagonally dominant.)

Suppose the initial approximation be;

$x_1 = a$

$x_2 = b$

$x_3 = c$

Substituting these values in above equation; we get,

$$x_1^{(1)} = \frac{b_1 - a_{12}b - a_{13}c}{a_{11}}$$

$$x_2^{(1)} = \frac{b_2 - a_{21}a - a_{23}c}{a_{22}}$$

$$x_3^{(1)} = \frac{b_3 - a_{31}a - a_{32}b}{a_{33}}$$

Substituting these values of $x_1^{(1)}$, $x_2^{(1)}$ & $x_3^{(1)}$ into the right hand side of eq. (i), we get the second approximation $x_1^{(2)}$, $x_2^{(2)}$ & $x_3^{(2)}$. In the same way, the third approximation ($x_1^{(3)}$, $x_2^{(3)}$ & $x_3^{(3)}$) is computed by substituting the second approximation's x-values into the right hand side of eq.(i). This process is continued until the difference between two consecutive approximation is negligible.

Examples

1. Solve the following system using Gauss Jacobi iteration method.

$$\mathbf{6x_1 - 2x_2 + x_3 = 11}$$

$$\mathbf{-2x_1 + 7x_2 + 2x_3 = 5}$$

$$\mathbf{x_1 + 2x_2 - 5x_3 = -1}$$

Sol^n:

Given system,

$$6x_1 - 2x_2 + x_3 = 11$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$
$$x_1 + 2x_2 - 5x_3 = -1$$

Now, the given system can be written as;

$$x_1 = \frac{11+2x_2-x_3}{6}$$

$$x_2 = \frac{5+2x_1-2x_3}{7}$$

$$x_3 = \frac{-1-x_1-2x_2}{-5}$$

Let initial approximation be;
$x_1 = 0, x_2 = 0, x_3 = 0$

Then, successive iteration is given as;

n	x_1	x_2	x_3
Initial	0	0	0
1	1.833	0.714	0.2
2	2.038	1.181	0.852
3	2.085	1.053	1.089
4	2.003	1.003	1.038
5	1.995	1.138	1.002
6	2.046	1.141	1.054
7	2.038	1.148	1.066
8	2.038	1.144	1.067
9	2.037	0.992	1.065
10	1.987	0.992	1.004
11	1.997	0.996	0.994
12	1.999	1.000	0.998
13	2.000	1.000	1.000
14	2.000	1.000	1.000

The iteration at 13^{th} and 14^{th} positions is similar.

Hence,
$x_1 = 2$
$x_2 = 1$
$x_3 = 1$

2. *Solve the following set of equations using Gauss Jacobi iteration method.*

$$\boldsymbol{x_1 + 10x_2 + x_3 = 24}$$

$$\boldsymbol{10x_1 + x_2 + x_3 = 15}$$

$$\boldsymbol{x_1 + x_2 + 10x_3 = 33}$$

Sol^n:

Given system,

$$x_1 + 10x_2 + x_3 = 24$$

$$10x_1 + x_2 + x_3 = 15$$

$$x_1 + x_2 + 10x_3 = 33$$

The coefficient matrix of the given system of equations is not diagonally dominant. By re-arranging the given system of equation & can be written in diagonally dominant form as;

$$10x_1 + x_2 + x_3 = 15$$

$$x_1 + 10x_2 + x_3 = 24$$

$$x_1 + x_2 + 10x_3 = 33$$

Now, the given system can be written as;

$$x_1 = \frac{15 - x_2 - x_3}{10}$$

$$x_2 = \frac{24 - x_1 - x_3}{10}$$

$$x_3 = \frac{33 - x_1 - x_2}{10}$$

Let initial approximation be;

$x_1 = 0, x_2 = 0, x_3 = 0$

Then, successive iteration is given as;

n	x_1	x_2	x_3
Initial	0	0	0
1	1.5	2.4	3.3
2	0.93	1.92	2.91
3	1.017	2.016	3.015
4	0.997	1.997	2.997
5	1.001	2.001	3.001
6	0.999	1.999	2.999
7	1.000	2.000	3.000
8	1.000	2.000	3.000

The value at 7th and 8th iteration is same. Hence,

$x_1 = 1$

$x_2 = 2$

$x_3 = 3$

➢ Gauss-Seidel Iterative Method

With the Jacobi method, the values of $x_i^{(k)}$ obtained in k^{th} iteration remain unchanged until the entire $(k+1)^{th}$ iteration has been calculated. With the **Gauss-Seidel method**, we use the new values $x_i^{(k+1)}$ as soon as they are known. For example;

From Gauss Jacobi approximation; we have, (with initial approximation $x_1 = a$, $x_2 = b$ & $x_3 = c$)

$$x_1 = \frac{b_1 - a_{12}b - a_{13}c}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}a - a_{23}c}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31}a - a_{32}b}{a_{33}}$$

And, from Gauss Seidel approximation;

$$x_1 = \frac{b_1 - a_{12}b - a_{13}c}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}c}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

The process is repeated till the values of x_1, x_2 & x_3 obtained to desired accuracy.

Examples

1. Solve the following system using Gauss Seidel iteration method.

$$\mathbf{10x_1 + 2x_2 + x_3 = 9}$$

$$\mathbf{2x_1 + 20x_2 - 2x_3 = -44}$$

$$\mathbf{-2x_1 + 3x_2 + 10x_3 = 22}$$

Sol^n:

Given system,

$$10x_1 + 2x_2 + x_3 = 9$$

$$2x_1 + 20x_2 - 2x_3 = -44$$

$$-2x_1 + 3x_2 + 10x_3 = 22$$

Now, the given system can be written as;

$$x_1 = \frac{9-2x_2-x_3}{10}$$

$$x_2 = \frac{-44-2x_1+2x_3}{20}$$

$$x_3 = \frac{22+2x_1-3x_2}{10}$$

Let initial approximation be;

$x_1 = 0, x_2 = 0, x_3 = 0$

Then, successive iteration is given as;

n	x_1	x_2	x_3
Initial	0	0	0
1	0.9	-2.29	3.067
2	1.0513	-1.9984	3.0098
3	0.9987	-1.9705	2.8441
4	0.99	-1.9999	2.9979
5	1	-2	3
6	1	-2	3

The iteration at 5^{th} and 6^{th} positions is similar.

Hence,

$x_1 = 1$

$x_2 = -2$

$x_3 = 3$

2. ***Solve the following set of equations using Gauss Seidel iteration method.***

$$\boldsymbol{x_1 + 10x_2 + x_3 = 24}$$

$$\boldsymbol{10x_1 + x_2 + x_3 = 15}$$

$$\boldsymbol{x_1 + x_2 + 10x_3 = 33}$$

Sol^n^:

Given system;

$x_1 + 10x_2 + x_3 = 24$

$10x_1 + x_2 + x_3 = 15$

$x_1 + x_2 + 10x_3 = 33$

The coefficient matrix of the given system of equations is not diagonally dominant. By re-arranging the given system of equation & can be written in diagonally dominant form as;

$$10x_1 + x_2 + x_3 = 15$$

$$x_1 + 10x_2 + x_3 = 24$$

$$x_1 + x_2 + 10x_3 = 33$$

Now, the given system can be written as;

$$x_1 = \frac{15 - x_2 - x_3}{10}$$

$$x_2 = \frac{24 - x_1 - x_3}{10}$$

$$x_3 = \frac{33 - x_1 - x_2}{10}$$

Let initial approximation be;

$x_1 = 0, x_2 = 0, x_3 = 0$

Then, successive iteration is given as;

n	x_1	x_2	x_3
Initial	0	0	0
1	1.5	2.25	2.925
2	0.982	2.009	3.000
3	0.999	2.000	3.000
4	1.000	2.000	3.000
5	1.000	2.000	3.000

The iteration at 4th and 5th positions is similar.

Hence,

$x_1 = 1$
$x_2 = 2$
$x_3 = 3$

Q. Compare and contrast between Jacobi iterative method and Gauss Seidel method.

Solution:

Both the Jacobi and Gauss-Seidel methods are iterative methods for solving the linear system Ax = b.

In the Jacobi method the updated vector x is used for the computations only after all the variables (i.e. all components of the vector x) have been updated. On the other hand in the Gauss-Seidel method, the updated variables are used in the computations as soon as they are updated.

Thus in the Jacobi method, during the computations for a particular iteration, the “known” values are all from the previous iteration. However in the Gauss-Seidel method, the “known” values are a mix of variable values from the previous iteration (whose values have not yet been evaluated in the current iteration), as well as variable values that have already been updated in the current iteration.

Even though the Gauss-Seidel’s method uses the improved values as soon as they are computed, this does not ensure that the Gauss-Seidel’s method would converge faster than Jacobi iterations.

➢ Eigen Value and Eigen Vector

If a 'A' is any square matrix of order $n \times n$ with elements a_{ij} we can find a column matrix 'X' and a constant λ such that;

$$AX = \lambda X$$

Where, λ is called Eigen value and X is called corresponding Eigen vector.

Eigen value and Eigen vector using Power Method:

Algorithm:

1. *Input matrix **A**, initial vector **X**, error tolerance(EPS) and maximum iteration permitted(MAXIT)*
2. *Compute **Y=AX***
3. *Find the element k of Y that is largest in magnitude*
4. *Compute **X= Y/k***
5. *If $|X\text{-}X_{old}|$<EPS or iterations>MAXIT*
   *Write **k** and **X***
   else
   Goto step 2

Examples

1. Find the largest Eigen value and corresponding vector of the following matrix using power method.

$$\begin{bmatrix} \mathbf{1} & \mathbf{6} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \end{bmatrix}$$

Sol^n:

Note: *For the initial approximation of Eigen vector take 1 for the rows not having zero and 0 for the other. If all rows contains 0 then take (0 1 0).*

Let the initial Eigen vector is;

$$X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

1st iteration:

$$AX^{(0)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

2^nd^ iteration:

$$AX^{(1)} = \begin{bmatrix}1 & 6 & 1\\1 & 2 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}7\\3\\0\end{bmatrix} = 7\begin{bmatrix}1\\0.429\\0\end{bmatrix} = \lambda^{(2)}X^{(2)}$$

Take largest value as common and divide the whole element of the vector with largest value.

3^rd^ iteration:

$$AX^{(2)} = \begin{bmatrix}1 & 6 & 1\\1 & 2 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}1\\0.429\\0\end{bmatrix} = \begin{bmatrix}3.571\\1.858\\0\end{bmatrix} = 3.571\begin{bmatrix}1\\0.520\\0\end{bmatrix} = \lambda^{(3)}X^{(3)}$$

4^th^ iteration:

$$AX^{(3)} = \begin{bmatrix}1 & 6 & 1\\1 & 2 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}1\\0.520\\0\end{bmatrix} = \begin{bmatrix}4.120\\2.040\\0\end{bmatrix} = 4.120\begin{bmatrix}1\\0.495\\0\end{bmatrix} = \lambda^{(4)}X^{(4)}$$

5^th^ iteration:

$$AX^{(4)} = \begin{bmatrix}1 & 6 & 1\\1 & 2 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}1\\0.495\\0\end{bmatrix} = \begin{bmatrix}3.971\\1.990\\0\end{bmatrix} = 3.971\begin{bmatrix}1\\0.501\\0\end{bmatrix} = \lambda^{(5)}X^{(5)}$$

Note: *Repeat this process till [$X^{(n)} - X^{(n-1)}$] becomes negligible, then $\lambda^{(n)}$ will be the largest eigen value and $X^{(n)}$ be the corresponding eigen vector.*

6^th^ iteration:

$$AX^{(5)} = \begin{bmatrix}1 & 6 & 1\\1 & 2 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}1\\0.501\\0\end{bmatrix} = \begin{bmatrix}4.006\\2.002\\0\end{bmatrix} = 4.006\begin{bmatrix}1\\0.500\\0\end{bmatrix} = \lambda^{(6)}X^{(6)}$$

7^th^ iteration:

$$AX^{(6)} = \begin{bmatrix}1 & 6 & 1\\1 & 2 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}1\\0.500\\0\end{bmatrix} = \begin{bmatrix}4\\2\\0\end{bmatrix} = 4\begin{bmatrix}1\\0.5\\0\end{bmatrix} = \lambda^{(7)}X^{(7)}$$

∴ **Largest Eigen value = 4 and Eigen vector =** $\begin{bmatrix}\mathbf{1}\\\mathbf{0.5}\\\mathbf{0}\end{bmatrix}$

2. Find the largest Eigen value correct to two significant digits and corresponding Eigen vector of the following matrix using power method.

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 3 \end{bmatrix}$$

Sol^n^:

Let the initial Eigen vector is;

$$X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

1^st^ iteration:

$$AX^{(0)} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

2^nd^ iteration:

$$AX^{(1)} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1.5 \\ 2.5 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 0.6 \\ 1 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

3^rd^ iteration:

$$AX^{(2)} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.6 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.4 \\ 3.6 \\ 4 \end{bmatrix} = 5.4 \begin{bmatrix} 1 \\ 0.67 \\ 0.74 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

4^th^ iteration:

$$AX^{(3)} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.67 \\ 0.74 \end{bmatrix} = \begin{bmatrix} 5.42 \\ 2.89 \\ 3.22 \end{bmatrix} = 5.42 \begin{bmatrix} 1 \\ 0.53 \\ 0.59 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

5^th^ iteration:

$$AX^{(4)} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.53 \\ 0.59 \end{bmatrix} = \begin{bmatrix} 4.71 \\ 2.3 \\ 2.77 \end{bmatrix} = 4.71 \begin{bmatrix} 1 \\ 0.49 \\ 0.60 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

6^th^ iteration:

$$AX^{(5)} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.49 \\ 0.59 \end{bmatrix} = \begin{bmatrix} 4.55 \\ 2.26 \\ 2.77 \end{bmatrix} = 4.55 \begin{bmatrix} 1 \\ 0.49 \\ 0.60 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

∴ Largest Eigen value = 4.55 and,

$$\text{Eigen vector} = \begin{bmatrix} 1 \\ 0.49 \\ 0.60 \end{bmatrix}$$

References:

- *E. Balagurusamy, Numerical Methods, Tata McGraw-Hill*

Please let me know if I missed anything or anything is incorrect.

Poudeljayanta99@gmail.com