Unit 5

Solution of Ordinary Differential Equations

Let x be an independent variable and y be a dependent variable. An equation with x, y and its derivatives is called a differential equation.

Suppose the first order differential equation;

$$\frac{dy}{dx} = f(x, y) \quad \ldots\ldots\ldots.. \quad (i)$$

A solution to the differential equation is the value of y which satisfies the differential equation.

Initial Value Problem

Consider the differential equation

$y' = f(x, y)$ with an initial condition $y(x_0) = y_0$.

This is the first order differential equation. Here the y value at x_0 is given to be y_0. The solution y at x_0 is given.

We must assume a small increment h.

$$x_1 = x_0 + h$$

$$x_2 = x_1 + h$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$x_{i+1} = x_i + h$$

Let us denote the y values at $x_1, x_2, \ldots\ldots$ as $y_1, y_2, \ldots\ldots.$ respectively.

y_0 is given and we must find out $y_1, y_2, \ldots\ldots.$

The initial value y_0 is given. So, this differential equation is called an ***initial value problem.***

➢ Taylor's Series Method

y is a function of x. It is written as $y(x)$. By Taylor's series about the point x_0;

$$y(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots\ldots\ldots$$

x_0 & y_0 denote the initial value of x & y.

Examples

1. Find by Taylor's series method, the values of y at $x = 0.1$ & $x = 0.2$ to fine places of decimal form.

$\frac{dy}{dx} = x^2y - 1; \quad y(0) = 1$

Solution:

Given,

$\frac{dy}{dx} = x^2y - 1$

$y(0) = 1$

i.e. $x_0 = 0$ & $y_0 = 1$

Here,

$y' = x^2y - 1$

$y'' = x^2y' + 2xy$

$y''' = x^2y'' + 2xy' + 2(xy' + y) = x^2y'' + 4xy' + 2y$

$y^{iv} = x^2y''' + 6xy'' + 6y'$

Now at $x_0 = 0$ & $y_0 = 1$;

$y_0' = x_0^2y_0 - 1 = 0 - 1 = -1$

$y_0'' = x_0^2y_0' + 2x_0y_0 = 0 + 0 = 0$

$y_0''' = x_0^2y_0'' + 4x_0y_0' + 2y_0 = 0 + 0 + 2 \times 1 = 2$

$y_0^{iv} = x_0^2y_0''' + 6x_0y_0'' + 6y_0' = 0 + 0 + 6 \times (-1) = -6$

Now, the Taylor's series is;

$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0''''$ [Neglecting higher term]

$= 1 + \frac{x}{1!} \times (-1) + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times 2 + \frac{x^4}{4!} \times (-6)$

$= 1 - \frac{x}{1!} + \frac{2x^3}{3!} - \frac{6x^4}{4!}$

$\therefore y(0.1) = 1 - \frac{0.1}{1!} + \frac{2\times0.1^3}{3!} - \frac{6\times0.1^4}{4!} = 0.900308$

$\therefore y(0.2) = 1 - \frac{0.2}{1!} + \frac{2\times0.2^3}{3!} - \frac{6\times0.2^4}{4!} = 0.802267$

2. *Find the solution of following differential equation using Taylor's series method.*

$$y' = (x^3 + xy^2)e^{(-x)},\ y(0) = 1 \text{ to find } y \text{ at } x = 0.1, 0.2, 0.3$$

Sol^n:

Given,

$y' = (x^3 + xy^2)e^{(-x)}$

$y(0) = 1$ i.e. $x_0 = 0$ & $y_0 = 1$

Here

$y' = (x^3 + xy^2)e^{(-x)}$

$$\begin{aligned} y'' &= (x^3 + xy^2)(-e^{-x}) + e^{-x}(3x^2 + x2yy' + y^2) \\ &= e^{-x}(3x^2 + x2yy' + y^2 - x^3 - xy^2) \\ &= e^{-x}(3x^2 - x^3 + y^2 + 2xyy' - xy^2) \end{aligned}$$

$$\begin{aligned} y''' &= e^{-x}[6x - 3x^2 + 2yy' + 2\{x(yy'' + (y')^2) + yy'\} - (x2yy' + y^2)] - e^{-x}(3x^2 - x^3 + y^2 + 2xyy' - xy^2) \\ &= e^{-x}[6x - 3x^2 + 2yy' + 2x(yy'' + (y')^2) + 2yy' - 2xyy' - y^2 - 3x^2 + x^3 - y^2 - 2xyy' + xy^2] \\ &= e^{-x}[x^3 - 6x^2 + 6x + 4yy' + 2x(yy'' + (y')^2) - 4xyy' - 2y^2 + xy^2] \end{aligned}$$

Now at $x_0 = 0$ & $y_0 = 1$;

$y_0' = 0$

$y_0'' = 1(0 - 0 + 1 + 0 - 0) = 1$

$y_0''' = 1(0 - 0 + 0 + 0 + 0 - 0 - 2 + 0) = -2$

Now, the Taylor's series is;

$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0'''$ [Neglecting higher term]

$= 1 + \frac{(x-0)}{1!}(0) + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(-2)$

$= 1 + \frac{x^2}{2} - \frac{x^3}{3}$

$\therefore y(0.1) = 1 + \frac{(0.1)^2}{2} - \frac{(0.1)^3}{3} = 1.0047$

$\therefore y(0.2) = 1 + \frac{(0.2)^2}{2} - \frac{(0.2)^3}{3} = 1.0173$

$\therefore y(0.3) = 1 + \frac{(0.3)^2}{2} - \frac{(0.3)^3}{3} = 1.036$

3. *Use the Taylor method to solve the equation*

$$\boldsymbol{y' = x^2 + y^2}$$

for $\boldsymbol{x = 0.25}$ ***and*** $\boldsymbol{x = 0.5}$ ***given*** $\boldsymbol{y(0) = 1.}$

Sol^n:

Given,

$y' = x^2 + y^2$

$y(0) = 1$ i.e. $x_0 = 0$ & $y_0 = 1$

Here,

$y' = x^2 + y^2$

$y'' = 2x + 2yy'$

$y''' = 2 + 2yy'' + 2(y')^2$

Now at $x_0 = 0$ & $y_0 = 1$;

$y_0' = 1$

$y_0'' = 0 + 2 = 2$

$y_0''' = 2 + 4 + 2 = 8$

Now, the Taylor's series is;

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' \quad \text{[Neglecting higher term]}$$

$$= 1 + \frac{(x-0)}{1!}(1) + \frac{(x-0)^2}{2!}(2) + \frac{(x-0)^3}{3!}(8)$$

$$= 1 + x + x^2 + \frac{8x^3}{3!}$$

$$\therefore y(0.25) = 1 + 0.25 + (0.25)^2 + \frac{8(0.25)^3}{3!} = 1.3333$$

$$\therefore y(0.25) = 1 + 0.5 + (0.5)^2 + \frac{8(0.5)^3}{3!} = 1.81667$$

- **Picard's Method**

Consider the differential equation;

$$\frac{dy}{dx} = f(x,y) \;\ldots\ldots\ldots\; (i)$$

with given initial condition $y(x_0) = y_0$

Eq.(i) can be written as

$$dy = f(x,y)dx \text{ ……….. (ii)}$$

Integrating eq.(ii) from x_0 to x w.r.to x.

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)dx$$

$$[y]_{y_0}^{y} = \int_{x_0}^{x} f(x,y)dx$$

$$y - y_0 = \int_{x_0}^{x} f(x,y)dx$$

$$y = y_0 + \int_{x_0}^{x} f(x,y)dx$$

For 1st approximation we replace y by y_0 we get,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

For 2nd approximation we replace y by y_1 we get,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

Similarly, for other approximation we make a general form;

$$y_i = y_0 + \int_{x_0}^{x} f(x, y_{i-1})dx$$

We continue this process until we get two successive approximation value equal.

Examples

1. ***Obtain a solution up to the fifth approximation of the equation*** $\frac{dy}{dx} = y + x$ ***such that*** $y(0) = 1$ ***using Picard's process of successive approximation.***

Solution:

Here,

$\frac{dy}{dx} = y + x$

$y(0) = 1$

i.e. $x_0 = 0$ & $y_0 = 1$

Using Picard's formula, we have;

$$y_i = y_0 + \int_{x_0}^{x} f(x, y_{i-1})dx$$

1st approximation;

$$y_1 = y_0 + \int_0^x f(x, y_0)dx$$
$$y_1 = 1 + \int_0^x (y_0 + x)dx$$
$$y_1 = 1 + \int_0^x (y_0 + x)dx$$
$$y_1 = 1 + \int_0^x (1 + x)dx$$
$$y_1 = 1 + x + \frac{x^2}{2}$$

2nd approximation;

$$y_2 = y_0 + \int_0^x f(x, y_1)dx$$
$$y_2 = 1 + \int_0^x (y_1 + x)dx$$
$$y_2 = 1 + \int_0^x (1 + 2x + \frac{x^2}{2})dx$$
$$y_2 = 1 + x + x^2 + \frac{x^3}{6}$$

3rd approximation;

$$y_3 = y_0 + \int_0^x f(x, y_2)dx$$
$$y_3 = 1 + \int_0^x (y_2 + x)dx$$
$$y_3 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{6})dx$$
$$y_3 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

4th approximation;

$$y_4 = y_0 + \int_0^x f(x, y_3)dx$$
$$y_4 = 1 + \int_0^x (y_3 + x)dx$$
$$y_4 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24})dx$$
$$y_4 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

5th approximation;

$$y_5 = y_0 + \int_0^x f(x, y_4)dx$$
$$y_4 = 1 + \int_0^x (y_4 + x)dx$$
$$y_4 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120})dx$$
$$y_4 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120})dx$$
$$y_4 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}$$

2. *Use Picard's method, estimate y(0.1) of the following equation;*

$$y'(x) = x^2 + y^2, \quad y(0) = 0$$

Solution:

Here,

$y'(x) = x^2 + y^2$

$y(0) = 0$

i.e. $x_0 = 0$ & $y_0 = 0$

Using Picard's formula, we have;

$$y_i = y_0 + \int_{x_0}^{x} f(x, y_{i-1})dx$$

First approximation;

$y_1 = y_0 + \int_0^x f(x, y_0)dx$

$y_1 = 0 + \int_0^x x^2 dx = \frac{x^3}{3}$

$At\ x = 0.1,$

$y_1 = 0.00033$

Second approximation,

$y_2 = y_0 + \int_0^x f(x, y_1)dx$

$y_2 = 0 + \int_0^x f(x, y_1)dx = \frac{x^3}{3} + \frac{x^7}{63}$

$At\ x = 0.1,$

$y_2 = 0.00033$

Here, $y_1 = y_2$ up to 5 decimal places.

$\therefore y(0.1) = 0.00033$

- **Euler's Method**

In Euler's method, the slope at (x_i, y_i) is used to estimate the value of $y(x_{i+1})$ as below;

$$y(x_{i+1}) = y(x_i) + m_1 h \ ; \ m_1 = f(x_i, y_i)$$

Choosing smaller values of h leads to more accurate results and more computation time.

Algorithm:

1. *Define* $f(x, y)$.
2. *Read* x_0, y_0, h *and* xn *where* x_0& y_0 *are initial conditions, h is the interval and xp is the required value.*
3. $n = \frac{xp - x_0}{h}$
4. *Start loop from* $i = 1$ *to* n
5. $y = y_0 + h * f(x_0, y_0)$
   $x = x + h$
6. *Print values of* y_0 & x_0.
7. *Check if* $x < xp$
   assign $x_0 = x$ *and* $y_0 = y$
   else
   goto 8.
8. *End loop i*
9. *Stop*

Examples

1. *Given* $\boldsymbol{y' = xy}$, $\boldsymbol{y(1) = 1}$. ***Find*** $\boldsymbol{y(2)}$ ***with*** $\boldsymbol{h = 0.25}$.

Solution:

Here,
$y' = f(x, y) = xy$

$y(1) = 1$ **i.e.** $x_0 = 1$ & $y_0 = 1$

Then,
$y(1) = y_0 = 1$

$y(1.25) = y_1 = y_0 + hf(x_0, y_0) = y_0 + h(x_0 * y_0) = 1 + 0.25 * (1 * 1) = 1.25$

$y(1.5) = y_2 = y_1 + hf(x_1, y_1) = 1.25 + 0.25 * (1.25 * 1.25) = 1.64$

$y(1.75) = y_3 = y_2 + hf(x_2, y_2) = 1.64 + 0.25 * (1.5 * 1.64) = 2.26$

$y(2) = y_4 = y_3 + hf(x_3, y_3) = 2.26 + 0.25 * (1.75 * 2.26) = 3.25$

Hence,
$y(2) = 3.25$

2. *Given the equation $y' = 2x^3 - 3xy$, $y(1) = 2$. Find $y(2.5)$ with $h = 0.5$.*

Solution:

Here,
$y' = f(x, y) = 2x^3 - 3xy$

$y(1) = 2$ **i.e.** $x_0 = 1$ & $y_0 = 2$

Then,
$y(1) = y_0 = 2$

$y(1.5) = y_1 = y_0 + hf(x_0, y_0) = 2 + 0.5[2 - 3(1)(2)] = 0$

$y(2) = y_2 = y_1 + hf(x_1, y_1) = 0 + 0.5[2 * 1.5^3 - 3 * 1.5 * 0] = 3.375$

$y(2.5) = y_3 = y_2 + hf(x_2, y_2) = 3.375 + 0.5[2 * 2^3 - 3 * 2 * 3.375] = 1.25$

Hence,
$y(2.5) = 1.25$

- **Heun's Method**

- This method is also called ***second order Runge-Kutta method*** or ***Modified Euler's method.***

In Heun's method, we use the average of the slopes computed at the beginning and at the end of the interval.
Using Heun's method, we can estimate the value of $y(x_{i+1})$ as below;

$$y(x_{i+1}) = y(x_i) + \frac{h}{2}(m_1 + m_2) \qquad // \; y(x_{i+1}) = y(x_i + h)$$

Where, $m_1 = f(x_i, y_i)$
$m_2 = f(x_i + h, y_i + m_1 \times h)$

Algorithm:

1. *Define* $f(x, y)$.
2. *Read* x_0, y_0, h *and* n
3. *For i=0 to n-1 do*
4. $x_{i+1} = x_i + h$
5. $m_1 = f(x_i, y_i)$
6. $m_2 = f(x_i + h, y_i + m_1 \times h)$
7. $y_{i+1} = y_i + \frac{h}{2}(m_1 + m_2)$
8. *Print* x_{i+1} , y_{i+1}
9. *Next i*
10. *End*

Examples

1. Use the Heun's method to estimate $y(0.4)$ when $y'(x) = x^2 + y^2$ with $y(0) = 0$. Assume $h = 0.2$.

Solution:

Here,
$y'(x) = f(x, y) = x^2 + y^2$
$y(0) = 0$ **i.e.** $x_0 = 0$ & $y_0 = 0$
$h = 0.2$
From Heun's method, we have;

1st iteration:

$$m_1 = f(x_0, y_0) = x_0^2 + y_0^2 = 0 + 0 = 0$$

$$m_2 = f(x_0 + h, y_0 + m_1 * h) = f(0 + 0.2, 0 + 0 * 0.2) = f(0.2, 0) = 0.2^2 + 0^2 = 0.04$$

$$\therefore y(x_0 + h) = y(x_0) + \frac{h}{2}(m_1 + m_2)$$

$$y(0 + 0.2) = y(0.2) = y(0) + \frac{0.2}{2}(m_1 + m_2) = 0 + \frac{0.2}{2}(0 + 0.04) = 0.004$$

$$\therefore y(0.2) = 0.004$$

2nd iteration:
Here,
$x_1 = 0.2$ & $y_1 = 0.004$

$$m_1 = f(x_1, y_1) = x_1^2 + y_1^2 = 0.2^2 + 0.004^2 = 0.040016$$

$$m_2 = f(x_1 + h, y_1 + m_1 * h) = f(0.4, 0.012) = 0.4^2 + 0.012^2 = 0.160144$$

$$\therefore y(x_1 + h) = y(x_1) + \frac{h}{2}(m_1 + m_2)$$

$$y(0.2 + 0.2) = y(0.4) = y(0.2) + \frac{0.2}{2}(0.040016 + 0.160144) = 0.004 + 0.02 = 0.024$$

$$\therefore \mathbf{y(0.4) = 0.024}$$

2. Apply Runge Kutta method of 2nd order to find an approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x + y$ and $y(0) = 1$.

Solution:

Here,
$\frac{dy}{dx} = x + y$
$y(0) = 1$ i.e. $x_0 = 0$ & $y_0 = 1$
let us assume $h = 0.2$

$$m_1 = f(x_0, y_0) = f(0, 1) = 0 + 1 = 1$$

$$m_2 = f(x_0 + h, y_0 + m_1 \times h) = f(0 + 0.2, 1 + 1 \times 0.2) = f(0.2, 1.2) = 1.4$$

Then,

$y(0.2) = y(0) + \frac{0.2}{2}(m_1 + m_2) = 1 + \frac{0.2}{2}(1 + 1.4) = 1.24$

$\therefore y(0.2) = 1.24$

- **Fourth Order Runge-Kutta (R-K) Method**

$$y_{i+1} = y_i + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

Where,

$$m_1 = f(x_i, y_i)$$
$$m_2 = f(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2})$$
$$m_3 = f(x_i + \frac{h}{2}, y_i + \frac{m_2 h}{2})$$
$$m_4 = f(x_i + h, y_i + m_3 h)$$

$$y_1 = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

Where,

$$m_1 = f(x_0, y_0)$$
$$m_2 = f(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2})$$
$$m_3 = f(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2})$$
$$m_4 = f(x_0 + h, y_0 + m_3 h)$$

Similarly, for second interval

$$y_2 = y_1 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

Where,

$$m_1 = f(x_1, y_1)$$
$$m_2 = f(x_1 + \frac{h}{2}, y_1 + \frac{m_1 h}{2})$$
$$m_3 = f(x_1 + \frac{h}{2}, y_1 + \frac{m_2 h}{2})$$
$$m_4 = f(x_1 + h, y_1 + m_3 h)$$

Algorithm:

1. *Define* $f(x, y)$.
2. *Read* x_0, y_0, h and n
3. *For i=0 to n-1 do*
4. $x_{i+1} = x_i + h$
5. $m_1 = f(x_i, y_i)$
6. $m_2 = f(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2})$
7. $m_3 = f(x_i + \frac{h}{2}, y_i + \frac{m_2 h}{2})$
8. $m_4 = f(x_i + h, y_i + m_3 h)$
9. $y_{i+1} = y_i + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$
10. *Print* x_{i+1} , y_{i+1}
11. *Next i*
12. *End*

Examples

1. Apply Runge Kutta method of 4^th order to find an approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x + y$ and $y(0) = 1$.

Solution:

Here,

$\frac{dy}{dx} = f(x, y) = x + y$

$y(0) = 1$ *i.e.* $x_0 = 0$ & $y_0 = 1$

let us assume $h = 0.2$

Now, from Runge-Kutta method, we have,

$$m_1 = f(x_0, y_0) = x_0 + y_0 = 0 + 1 = 1$$

$$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}\right) = f\left(0 + \frac{0.2}{2}, 1 + \frac{1 \times 0.2}{2}\right) = f(0.1, 1.1) = 1.2$$

$$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}\right) = f\left(0 + \frac{0.2}{2}, 1 + \frac{1.2 \times 0.2}{2}\right) = f(0.1, 1.12) = 1.22$$

$$m_4 = f(x_0 + h, y_0 + m_3 h) = f(0 + 0.2, 1 + 1.22 \times 0.2) = f(0.2, 1.244) = 1.444$$

Hence,

$$y_1 = y(x_0 + h) = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$\therefore y(0.2) = 1 + \frac{0.2}{6}(1 + 2 \times 1.2 + 2 \times 1.22 + 1.444)$$

$$= 1.2428$$

2. *Obtain y(1.5) from given differential equation using Runge-Kutta 4^th order method.*

$\frac{dy}{dx} + 2x^2y = 1$ ***and*** $y(1) = 0.$ ***[take h=0.25]***

Solution:

Here,

$f(x,y) = \frac{dy}{dx} = 1 - 2x^2y$

$y(1) = 0$ i.e. $x_0 = 1$ & $y_0 = 0$

h=0.25

Now, from Runge-Kutta method, we have,

1st iteration

$m_1 = f(x_0, y_0) = 1 - 2x_0^2y_0 = 1 - 2 \times 1^2 \times 0 = 1 - 0 = 1$

$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1h}{2}\right) = f\left(1 + \frac{0.25}{2}, 0 + \frac{1\times0.25}{2}\right) = f(1.125, 0.125) = 0.684$

$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2h}{2}\right) = f\left(1 + \frac{0.25}{2}, 0 + \frac{0.684\times0.25}{2}\right) = f(1.125, 0.0855) = 0.784$

$m_4 = f(x_0 + h, y_0 + m_3h) = f(1 + 0.25, 0 + 0.784 \times 0.25) = f(1.25, 0.196) = 0.388$

Hence,

$y_1 = y(x_0 + h) = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$

$\therefore y(1.25) = 0 + \frac{0.25}{6}(1 + 2 \times 0.684 + 2 \times 0.784 + 0.388)$

$= 0.18$

2nd iteration

Now,

$x_1 = 1.25$ & $y_1 = 0.18$

$m_1 = f(x_1, y_1) = 1 - 2x_1^2y_1 = 1 - 2 \times 1.25^2 \times 0.18 = 1 - 0.5625 = 0.437$

$m_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_1h}{2}\right) = f\left(1.25 + \frac{0.25}{2}, 0.18 + \frac{0.437\times0.25}{2}\right) = f(1.375, 0.235) = 0.111$

$m_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_2h}{2}\right) = f\left(1.25 + \frac{0.25}{2}, 0.18 + \frac{0.111\times0.25}{2}\right) = f(1.375, 0.194) = 0.266$

$m_4 = f(x_1 + h, y_1 + m_3h) = f(1.25 + 0.25, 0.18 + 0.266 \times 0.25) = f(1.5, 0.246) = -0.107$

Hence,

$y_2 = y(x_1 + h) = y_1 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$

$\therefore y(1.5) = 0.18 + \frac{0.25}{6}(0.437 + 2 \times 0.111 + 2 \times 0.266 + (-0.107))$

$= 0.2251$

$\therefore \boldsymbol{y(1.5) = 0.2251}$

Solving Higher Order Differential Equation

A high order differential equation is in the form

$$\frac{d^m y}{dx^m} = f(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\dots\dots, \frac{d^{m-1}y}{dx^{m-1}})$$

with m initial condition given as;

$y(x_0) = a_1$

$y'(x_0) = a_2$

. .

. .

. .

$y^{m-1}(x_0) = a_m$

Let us denote,

$y = y_1$

$\frac{dy}{dx} = y_2$

$\frac{d^2y}{dx^2} = y_3$

. .

. .

. .

$\frac{d^{m-1}y}{dx^{m-1}} = y_m$

Then we can write,

$\frac{dy_1}{dx} = y_2$ with $y_1(x_0) = y_{10} = a_1$

$\frac{dy_2}{dx} = y_3$ with $y_2(x_0) = y_{20} = a_2$

.

.

.

$\frac{dy^{m-1}}{dx} = y_m$ with $y_{m-1}(x_0) = y_{(m-1)0} = a_{m-1}$

$\frac{dy^m}{dx} = \frac{d^m y}{dx^m} = F(x, y_1, y_2, \dots\dots\dots, y_m)$ with $y_m(x_0) = y_{m0} = a_m$

This system is similar to the system of first order equation.

Hence, we can solve this by any procedure applied for first order equation.

Representation of Higher Order Equation into Simultaneous Equation

Consider the seconder order differential equation

$$y'' = f(x, y, y')$$
$$y(x_0) = y_0, y'(x_0) = y'_0$$

This can be converted into a system of Simultaneous equations.

Put $z = y'$

Therefore the equation becomes $z' = f(x, y, z)$

$$y(x_0) = y_0$$

That is we have,

$$y' = z$$
$$z' = f(x, y, z)$$
$$y(x_0) = y_0, z(x_0) = y_0$$

This is a set of Simultaneous equations and hence can be solved. Any higher order equation can thus be transformed into simultaneous equation.

Q. ***Solve the following differential equation for y(0.5) using Heun's method.***

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2xy = 1 \text{ with } y(0) = 1 \text{ and } y'(0) = 1\ .$$

Solution:

Here,

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2xy = 1$$

$$x_0 = 0 \;\&\; y_0 = 1, y'(0) = 1 = z_0$$

Put $\frac{dy}{dx} = z$ & differentiating w.r.to x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Equation assumes the form:

$$\frac{dz}{dx} + 3z + 2xy = 1$$

We have system of equations,

$$y' = \frac{dy}{dx} = z = f(x, y, z) \qquad \text{[let slope} = m_i\text{]}$$

$$\frac{dz}{dx} = 1 - 2xy - 3z = g(x, y, z) \qquad \text{[let slope} = l_i\text{]}$$

$let\ h = 0.5$

Now,

$$m_1 = f(x_0, y_0, z_0) = f(0,1,1) = 1$$

$$l_1 = g(x_0, y_0, z_0) = g(0,1,1) = 1 - 2 \times 0 \times 1 - 3 \times 1 = -2$$

Similarly,

$$m_2 = f(x_0 + h, y_0 + hm_1, z_0 + hl_1) = f(0 + 0.5, 1 + 0.5 \times 1, 1 + 0.5 \times (-2))$$

$$= f(0.5, 1.5, 0) = 0$$

$$l_2 = g(0.5, 1.5, 0) = 1 - 2 \times 0.5 \times 1.5 - 3 \times 0 = -0.5$$

$\therefore y(0.5) = y_0 + \frac{h}{2}(m_1 + m_2)$

$= 1 + \frac{0.5}{2}(1 + 0)$

$= 1.25$

$\therefore y'(0.5) = y'(0) + \frac{h}{2}(l_1 + l_2)$

$= 1 + \frac{0.5}{2}(-2 - 0.5)$

$= 0.375$

Q. ***Solve the following differential equation to find $y(0.1)$ using 4th order Runge-Kutta method.***

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1 \textit{ with } y(0) = 1 \textit{ and } y'(0) = 0$$

Solution:

Here,

$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$

$x_0 = 0 \;\&\; y_0 = 1, y'(0) = 0 = z_0$

Put $\frac{dy}{dx} = z$ & differentiating w.r.to x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Equation assumes the form:

$$\frac{dz}{dx} - x^2 z - 2xy = 1$$

We have system of equations,

$y' = \frac{dy}{dx} = z = f(x, y, z)$ [let slope = m_i]

$\frac{dz}{dx} = 1 + 2xy + x^2 z = g(x, y, z)$ [let slope = l_i]

We have,

$y(x_0 + h) = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$ ……….. (i)

$x_0 + h = 0.1 \Rightarrow h = 0.1$

$m_1 = f(x_0, y_0, z_0) = f(0, 1, 0) = 0$

$l_1 = g(x_0, y_0, z_0) = g(0, 1, 0) = 1 + 2 \times 0 \times 1 + 0^2 \times 0 = 1$

$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}, z_0 + \frac{l_1 h}{2}\right) = f(0.05, 1, 0.05) = 0.05$

$l_2 = g\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}, z_0 + \frac{l_1 h}{2}\right) = g(0.05, 1, 0.05) = 1.10$

$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}, z_0 + \frac{l_2 h}{2}\right) = f(0.05, 1.0025, 0.055) = 0.055$

$l_3 = g\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}, z_0 + \frac{l_2 h}{2}\right) = g(0.05, 1.0025, 0.055) = 1.1$

$m_4 = f(x_0 + h, y_0 + m_3 h, z_0 + l_3 h) = f(0.1, 1.005, 0.11) = 0.11$

$l_4 = g(x_0 + h, y_0 + m_3 h, z_0 + l_3 h) = g(0.1, 1.005, 0.11) = 1.202$

m_1, m_2, m_3 & m_4 values substituted in eq. (i)

$$y(0.1) = 1 + \frac{0.1}{6}(0 + 2 \times 0.05 + 2 \times 0.055 + 0.11)$$
$$= 1.0053$$

Boundary Value Problem

Consider the following linear second order differential equation,

$$y'' + f(x)y' + g(x)y = F(x)$$

Suppose we are interested in solving this differential equation between the values $x = a$ & $z = b$. Hence a & b are two values such that $a < b$. Let us divide the interval $[a, b]$ into n equal subintervals of length h each.

Let $x_0, x_1, \dots \dots \dots, x_n$ be the pivotal points and b are called the boundary points. Solving the differential equation means finding the values of $y_0, y_1, \dots \dots \dots, y_n$.

Suppose y_0 & y_n are given. That is the solution values at the boundary points are given. Then the differential equation is called a **Boundary Value Problem**. So, the following is the general form of a boundary value problem.

$$y'' + f(x)y' + g(x)y = F(x)$$

$$y(a) = y_0\,, \quad y(b) = y_n$$

References:

- *E. Balagurusamy, Numerical Methods, Tata McGraw-Hill*

Please let me know if I missed anything or anything is incorrect.

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